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BIAS CORRECTION FOR DYNAMIC FACTOR MODELS

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Keywords: *Dimensionality reduction; small sample bias correction; auto-regressive models; persistent process; Dynamic Factor Model.*

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Bias Correction for Dynamic Factor Models

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Abstract: In this paper we work with multivariate time series that follow a Dynamic Factor Model. In particular, we consider the setting where factors are dominated by highly persistent AutoRegressive (AR) processes, and samples that are rather small. Therefore, the factors' AR models are estimated using small sample bias correction techniques. A Monte Carlo study reveals that bias-correcting the AR coefficients of the factors allows to obtain better results in terms of prediction interval coverage. As expected, the simulation reveals that bias-correction is more successful for smaller samples. Results are gathered assuming the AR order and number of factors are known as well as unknown. We also study the advantages of this technique for a set of Industrial Production Indexes of several European countries.

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1. Introduction

Dimensionality reduction techniques have been employed for decades in the context of multiple time series datasets because "when the series are driven by a set of common factors, (a) a large number of parameters may be needed to obtain an adequate representation of the system and (b) the estimated parameters will be highly correlated. Therefore, a complex and badly defined relationship can appear when, in fact, a simpler and parsimonious model in terms of a few common factors can be operating." (Peña and Box, 1987).

This idea that large sets of time series can be modelled and forecast by using only a few variables that integrate information of all the data has been successfully applied in diverse research fields. Some examples are:

- Commodities' prices: Peña and Box (1987) extract common factors from wheat prices of different regions, Alonso et al. (2011), García-Martos et al. (2012), and Alonso et al. (2016) use dynamic factors for electricity prices, García-Martos et al. (2013) employ dynamic factors to model the volatility of electricity, fuel, and CO_2;
- Macroeconomic variables: Stock and Watson (2002) apply principal components to 149 economic indicators, Sargent and Sims (1977) work with index models in the context of business cycles, Forni et al. (1999) combine dynamic principal components and dynamic factor analysis to estimate economic activity indexes;
- Demographic variables: Lee and Carter (1992) use singular value decomposition to estimate indexes that help forecast age specific mortality in the US, Alonso et al. (2008) employ a dynamic factor model for mortality and fertility rates of the Spanish population.

In an iterative process, we can obtain forecasts for common factors, that allow to obtain forecasts for the original dataset or for other dependent variables. To obtain these forecasts, the common factors are modelled to follow, for instance, ARIMA models (Jeong and Bienkiewicz, 1997; García-Ferrer et al., 2011; García-Martos et al., 2012) or VAR and VARIMA models (Stock and Watson, 2002; Alonso et al., 2011; Peña and Poncela, 2006).

In this work we focus on modelling the common factors by means of AR models because we want to study one feature in particular: small sample bias-correction of nearly non-stationary AR coefficients. This aspect has been explored, among others, by Clements and Kim (2007), Roy and Fuller (2001), and Clements and Taylor (2001). However, to the best of our knowledge, it has not been studied in the context of factor models, even though it is not unusual to find highly persistent common factors. Some examples of this can be found in Alonso et al. (2011), for electricity prices of the Spanish market; and Gregory and Head (1999), for macroeconomic relations between investment, productivity and the current account in a multi-country setting. A simulation exercise will show that the improvements of employing the aforementioned corrections do not fade when the factors' forecasts are transformed back (through the estimated relation data series-factors) to become forecasts of the original time series. The proposal is illustrated with an empirical case as well, featuring the Industrial Production Index of several European countries.

The remainder of the paper is organized as follows. Section 2 describes the methodology. Section 3 presents the experimental design, and Section 4 shows the results of the Monte Carlo simulation. Section 5 contains an empirical application of the proposed methodology. Finally, Section 6 concludes.

2. Methodology

The methodology can be summarized in two steps. Given a vector of variables y_t, the first step consists of estimating the common factors. In the second step, an AR model for each factor is estimated. These AR models allow to obtain h-step-ahead (h being the forecasting horizon) forecasts[1] for the common factors, which, using the corresponding weights, are transformed to forecasts for y_t.

Though we work with AR models, it would be possible to extend the technique for seasonal ARIMA models. The complete process is described in the following subsections.

2.1. The Factor Model

As Geweke and Singleton (1981) explain, given an observable vector of time series, the factor model determines how many common factors there are; these factors can be interpreted as latent variables underlying the covariance structure of the series.

In the factor model, a set of observed variables, y_t, is decomposed into unobserved common factors, F_t, and specific components, ε_t. Let y_t be an N-dimensional observed vector of variables at time t, generated by an R-dimensional vector of unobserved common factors, with $R \ll N$. ε_t, the vector of specific components or idiosyncratic errors, is also N-dimensional. The factor model can be expressed as

$$y_t = \Omega F_t + \varepsilon_t, \tag{1}$$

where Ω is the matrix of loads or weights and has dimension $N \times R$. It indicates the relation of the R unobserved common factors with the observed series in y_t. The loadings in Ω are unknown, and we will consider only static weights (therefore, a static factor model). However, in a more general model, the effect of lagged factors may be included as well; in that case we would have a lagged polynomial matrix $\Omega(L)$ instead, where L is the lag operator. That is the so-called dynamic factor model. Bai and Ng (2008) indicate that for empirical applications the two approaches render similar forecasts, but the static approach, for which time domain methods are employed, is easier to estimate and implies fewer decisions regarding auxiliary parameters than the dynamic approach, which is estimated employing frequency domain analysis.

[1] See Marcellino et al. (2006) for a comparison between iterated multi-period ahead forecasts and direct forecasts for time series.

There are several techniques to estimate the unobserved common factors F_t. In their survey, Stock and Watson (2010) divide them in three groups: maximum likelihood by means of Kalman filter, non-parametric cross-sectional averaging, and hybrid techniques that combine the former two.

In this work we will use principal components (adapted to time series), which is included in the second set of methods. One advantage of this methodology is that it is computationally fast. Moreover, Stock and Watson (2002) prove that the factors' estimates obtained by means of principal components are consistent, even if there is serial or cross-sectional correlation in the specific components. Stock and Watson (2010) also indicate that when the number of variables is large, the estimation of the common factors is accurate enough that it can be included as data in regressions. We will be operating in a context like this: taking the cross-section dimension of the data to be high, while varying the length of the time dimension.

We obtain the common factors F_t by means of eigen-decomposition. This way we transform a matrix of data Y of size $(T \times N)$, where T represents the time dimension of the dataset, and N the cross sectional dimension, to a space with fewer dimensions, keeping those in which the data has the maximum variance. Let us recall the basics of this estimation: given $\Sigma_Y(N \times N)$ (in practice this will be the sample variance-covariance matrix for the dataset), we find real values λ and vectors e such that

$$\Sigma_Y e = \lambda e, \tag{2}$$

where λ are the so called eigenvalues of matrix Σ_Y, and e are the corresponding eigenvectors. When we do the multiplication in the left side, $\Sigma_Y e$, we are transforming the points of matrix Σ_Y into a new coordinates space. The objective is to keep only a few eigenvectors so that the transformation renders a space with fewer dimensions than the original dataset ($R < N$). As a property, assuming that the eigenvalues are different, the first component has greater variance than the second component, the second component has greater variance than the third component, and so on (Mardia et al., 1979, pp. 215).

This procedure is equivalent to minimizing a loss function given by the average squared difference between the data and the commonality, $y_t - \Omega F_t$, subject to a normalization and orthogonality of the weights (see Stock and Watson, 2010, for further details).

The estimation results in $\hat{\Omega}$ equal to a matrix of the eigenvectors of Σ_Y associated to the greatest R eigenvalues. Notice that it is infeasible to separately identify the common factors and their weights. Depending on the problem at hand, it is convenient to establish constrains, either for the factors or for the weights, that solve the identification problem. For reasons that will become clear in the simulations, we will constraint the weights to be orthogonal. For other details regarding the theory of factor models see Bai and Ng (2008).

This procedure for obtaining the factors is alike the dynamic extension (incorporates time dimension) of the principal components analysis (PCA) static case described in the appendix of García-Martos et al. (2012), and employed by Stock and Watson (2002). García-Martos et al. (2012) explain that, while Peña and Box (1987) dealt with stationary data, Lee and Carter (1992) employed non stationary data, suggesting that singular value decomposition (SVD) of the covariance matrix is used to compute the weights.

We are also proceeding similarly to Forni et al. (1999): employ principal components (PC) to separate the dynamics that create correlations in the whole panel, from the noise that characterizes each observed series and that is weakly related to the other observed series; and afterwards we incorporate the components in lieu of the factors in a dynamic factor model. However, since Forni et al. (1999) work with dynamic PC, they calculate the eigenvalues and eigenvectors of the spectral density matrix at different frequencies instead of those belonging to the data's covariance matrix.

It will not be an objective of this work to introduce methodology to model the idiosyncratic components. Therefore, we will take the specific factors ε_t as white noise. Furthermore, the specific factors' variances should be small in comparison to the variances of the common factors; otherwise they would be incorporated into the principal components (Mardia et al., 1979, pp.276).

Finally, we recur to criterion IC_3 of Bai and Ng (2002) to consistently estimate the number of factors R to keep in approximate factor models (meaning factor models in which the factors are approximated with PC). These authors define the criterion as $IC_3 = ln(V(r, \hat{F}^r)) + r(\frac{lnC_{NT}^2}{C_{NT}^2})$, where $V(r, \hat{F}^r)$ stands for the mean residual variance of employing r factors, and $\frac{lnC_{NT}^2}{C_{NT}^2}$ is the penalty for over-fitting. $C_{NT} = min(N, T)$, where N is number of time series included, and T is the series' length. Notice that we have changed their notation to be in line with the one hereby employed. Also, we use capital R to indicate the "true" (unknown) number of factors, and small r when referring to the estimated number of factors.

The advantage of this criterion is that it depends on both N and T, while other criteria such as the corrected Akaike Information Criterion (AICc, Hurvich and Tsai, 1989) or the Bayesian Information Criterion (BIC, Schwarz, 1978) only include one dimension (either N or T is taken as fixed). We selected IC_3 from the criteria proposed by Bai and Ng (2002) because it had better or equal performance than the others included in that paper, for values of N, T similar to the ones we employ in the simulation (see Tables I and II of Bai and Ng, 2002). We obtain an excellent performance of this criterion in our simulations, but another option would be to use Ahn and Horenstein (2013) test, which may work better in some circumstances.

2.2. AR factors

The factors F_t described in the previous section can be dynamic, following a time series model. We consider that each unobservable common factor $F_{i,t}$ is generated by an AR processes. Examples of AR common factors are given in Gregory and Head (1999) (study of the interactions of productivity, investment and current account), Fiorentini et al. (2016) (the authors estimate a single common factor for US sectoral employment data), Doz et al. (2012) (though the authors contemplate estimating VAR factors, in their simulation they generate AR factors), and García-Martos et al. (2013) (they estimate univariate GARCH models for common factors which represent volatility).

The following transition equation (Gregory and Head, 1999; Clements and Kim, 2007) describes each factor:

$$F_{i,t} = \phi_1 F_{i,t-1} + \phi_2 F_{i,t-2} + ... + \phi_p F_{i,t-p} + \eta_{i,t}. \tag{3}$$

We consider that the roots of the AR characteristic equation lie inside the unit circle (i.e. the process is stationary). We will specially pay attention to processes for which the factors are highly persistent, though not integrated, thus our focus is on close to unit roots of the characteristic polynomial of the AR model. For procedures that deal with integrated factors see Peña and Poncela (2006).

Also, $\eta_{i,t}$ will be normally distributed. However, in Appendix B we will see that this is not a restriction, and having other distribution for the errors $\eta_{i,t}$ does not alter the conclusions hereby obtained.

This particular type of model for the factors allows to maintain a low number of parameters to estimate. The AR coefficients are estimated by means of Conditional Sum of Squares (CSS) instead of Ordinary Least Squares (OLS) (Clements and Kim, 2007) in order to facilitate future extensions to include MA terms. In simulations, we assessed the estimates' distributions obtained by OLS and CSS for different values of the AR coefficients, and we observed that they overlap almost completely.

In order to select the number of lags, p, in each AR model, we will compare the performance of the AICc and BIC criteria. An alternative option not explored in this work would be to employ an endogenous lag order selection algorithm that re-estimates p in each iteration of the bootstrap (Kilian, 1998b).

2.3. *Small Sample Bias Correction*

CSS estimators for AR process are consistent.[2] However, in small samples some bias and skewness are often present. We employ two approaches in order to improve the estimation for highly persistent factors. On the one hand, the bootstrap bias-corrected estimator of Clements and Kim (2007), and on the other hand, the Roy-Fuller estimator (Roy and Fuller, 2001).

Clements and Kim (2007) bootstrap bias correction can be interpreted as a constant bias correction (MacKinnon and Smith, 1998); this means that the correction depends linearly on the value of the population parameter. This is a different approach from Roy and Fuller (2001) in that Roy-Fuller's estimate is mainly a function of the unit root test statistic.

To verify the accuracy of prediction intervals obtained based on these corrections, we will perform an extensive Monte Carlo experiment in Section 4.

2.3.1. Bootstrap Bias Correction

The procedure for Clements and Kim bootstrap bias correction may be summarized in the following steps. This description follows Clements and Kim (2007), and we adapt their notation to indicate that we are doing the correction in the models for the common factors, an extension of their procedure for a single series.

This process takes place after a first estimation of the AR model for each factor by means of CSS; we identify these coefficients as $\hat{\phi}_1, \hat{\phi}_2, ..., \hat{\phi}_p$. Notice there is a small difference of our approach with the one in Clements and Kim (2007); these authors use OLS in their estimations while we employ CSS in order to be able to incorporate MA terms in a future extension of this work. We take a shortcut and ignore sub-indexes for the factors since the procedure is the same for all of them.

1. Generate a pseudo-dataset f_t^* employing the estimated AR coefficients $\hat{\phi}_1, \hat{\phi}_2, ..., \hat{\phi}_p$, randomly selected residuals (η_t^*), and the first p estimates of the factor as starting values, $f_1, f_2, ..., f_p$.[3] We will repeat this step a number of times B.

$$f_t^* = \hat{\phi}_1 f_{t-1}^* + \hat{\phi}_2 f_{t-2}^* + ... + \hat{\phi}_p f_{t-p}^* + \eta_t^*. \tag{4}$$

2. Obtain the so called bootstrap estimates by re-estimating the AR coefficients for each pseudo-dataset $f_1^*, f_2^*, ..., f_T^*$ generated in the previous step. This means we will have B values $\hat{\phi}_1^*, \hat{\phi}_2^*, ..., \hat{\phi}_p^*$.
3. Clements and Kim (2007) explain that the bias can be estimated with the formula

$$bias = mean(\hat{\phi}^*) - \hat{\phi}. \tag{5}$$

   They obtain the bias-corrected estimator, $\hat{\phi}^{BC}$, by subtracting the bias from the OLS estimate (CSS for us instead here), and get

$$\hat{\phi}^{BC} = 2 \times \hat{\phi} - mean(\hat{\phi}^*). \tag{6}$$

4. Last, if needed, Kilian (1998a)'s algorithm is employed in order to adjust bootstrap estimates when they fall outside the stationary region. Any of the next three situations may arise:

   - When the original CSS estimates $\hat{\phi}$ are not stationary, we do not perform a bootstrap bias correction, thus $\hat{\phi}^{BC} = \hat{\phi}$.

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[2] Robinson (2006) shows that the CSS estimation converges a.s. in the context of long memory models. Also for long memory models, SARFIMA, Egrioglu et al. (2011) use simulation to show that CSS does better than a two-staged methodology.

[3] The notation for the common factors is in lowercase to emphasize that at this point we are working with factors that are estimates themselves; in other words, for each factor, $\hat{F} = f$.

- The corrected $\hat{\phi}^{BC}$ estimates should be used directly if they are stationary and the CSS $\hat{\phi}$ estimates are stationary as well.
- When the estimates of $\hat{\phi}$ are stationary but $\hat{\phi}^{BC}$ is not, then iterate i times until $\hat{\phi}_i^{BC}$ becomes stationary in the following way. We start with the values $\delta_1 = 1$, $\Delta_1 = bias$ (calculated in (5)), and calculate $\hat{\phi}_1^{BC} = \hat{\phi} - \Delta_1$. We will iterate i times, each time calculating $\Delta_{i+1} = \delta_i \Delta_i$, $\delta_{i+1} = \delta_i - 0.01$, $\hat{\phi}_i^{BC} = \hat{\phi} - \Delta_i$, until the estimates imply stationarity.

Kilian (1998a) shows that, because of small sample bias and skewness, bias-corrected bootstrap intervals are usually more accurate than the intervals obtained with other techniques, such as delta method, standard bootstrap, and Monte Carlo integration. This author works with bivariate models including VAR models, random walk models, and cointegrated processes, though not particularly with AR models like we do. Interestingly, Kilian (1998a) indicates that the procedure in step 4 does not have an effect asymptotically, and it is not constraining the OLS estimator because it affects the estimation of the bias and does not directly affect the OLS estimate.

2.3.2. Roy Fuller Estimator

As an alternative, we consider the estimator developed by Roy and Fuller (2001). The explanation in this section summarizes the relevant parts of that reference for this work. These authors' purpose is to obtain an estimator which provides with considerable gains in terms of mean square error for models that are close to the unit root, while maintaining a small loss in mean square error efficiency for the remainder parameter space. According to their simulations, the bias is reduced even if the process is not highly persistent.

Roy and Fuller (2001) start with a regression that works as an ARX (Auto Regressive Exogenous) with exogenous variables given by the lagged differences of the process, as it is done to test for a unit root in an AR(p) process. Since at this point we would be working with the estimated common factors, f, for our problem the regression would be

$$f_t = \hat{\theta}_1 f_{t-1} + \hat{\theta}_2 \Delta f_{t-1} + ... + \hat{\theta}_p \Delta f_{t-p+1} + u_t, \tag{7}$$

where $\hat{\theta}_1 = -\sum_{i=1}^{p} \hat{\phi}_i$, $\hat{\theta}_i = -\sum_{j=i}^{p} \hat{\phi}_j$, and $\Delta f_t = f_t - f_{t-1}$. Roy and Fuller (2001)'s correction depends on the LS estimator $\hat{\theta}_1$ (in our case estimated by CSS by adding up the auto-regressive coefficients $\hat{\phi}_i$), its standard error $\hat{\sigma}_1$, the unit root test statistic $\hat{\tau} = \frac{(\hat{\theta}_1 - 1)}{\hat{\sigma}_1}$, and a function C_p that corrects the bias and adapts depending on how close to the unit root the process is. Based on their paper, for us Roy-Fuller's corrected estimate would be,

$$\hat{\theta}_1^{RF} = \hat{\theta}_1^{CSS} + [C_p(\hat{\tau}) + C_{-p}(\hat{\tau})]\hat{\sigma}_1^{1/2}, \tag{8}$$

where the authors have established

$$C_p(\hat{\tau}) = \begin{cases} 0 & \text{for } \hat{\tau} \le -(k_1)^{1/2} \\ \lfloor \frac{p+1}{2} \rfloor n^{-1}\hat{\tau} - (s+1)\hat{\tau}^{-1} & \text{for } (k_1)^{1/2} < \hat{\tau} \le K \\ \lfloor \frac{p+1}{2} \rfloor n^{-1}\hat{\tau} - (s+1)(\hat{\tau} + k_2(\hat{\tau} - K))^{-1} & \text{for } K < \hat{\tau} \le \tau_{0.5} \\ -\tau_{0.5} + d_n(\hat{\tau} - \tau_{0.5}) & \text{for } \tau_{0.5} \le \hat{\tau}, \end{cases} \tag{9}$$

$k_1 = \lfloor (p+1)/2 \rfloor^{-1}(s+1)n$, $k_2 = [(1 + \lfloor (p+1)/2 \rfloor n^{-1})\tau_{0.5}(\tau_{0.5} - K)]^{-1}[(s+1) - \lfloor (p+1)/2 \rfloor n^{-1}\tau_{0.5}^2]$; $\tau_{0.5}$ is the median of τ when there is unit root; and d_n is a slope set to 0.1111 in Roy and Fuller (2001)'s simulations. Also $K = 5$, and s is the rank of exogenous explanatory variables (if any).

Functions $C_p(\hat{\tau})$ and $C_{-p}(\hat{\tau})$ are defined similarly. Clements and Kim (2007) indicate that $C_{-p}(\hat{\tau})$ is close to null for most time series employed in economics because these tend to have a unit root, or be close to unit root processes. See Roy and Fuller (2001) for details on this function.

2.4. Complete Process: Obtaining Forecasting Intervals

For calculating forecast intervals, we employ a bootstrap procedure based on Alonso et al. (2008). We follow the same steps, but we exclude estimation and forecasts of specific factors in the factor model, and we include bias corrections in the estimation of the AR coefficients.

We are using a parametric bootstrap, since we are estimating the model from the data only once, and then using this model as if it were the true model.[4] Ignoring model uncertainty will not be a problem when we specify in advance the value of p, known in simulations, but can definitely affect the estimation when p is unknown.

The process can be summarized in the next steps:

1. Using multiple series in a matrix, Y, we extract common factors by eigen-decomposition of the variance-covariance matrix.

Then, we conduct steps 2 to 4 separately for each extracted common factor.

2. Estimate an AR model for each factor. This involves two steps: first, selecting $\hat{p}$, and then estimating $\hat{\phi}_0, \hat{\phi}_1, ..., \hat{\phi}_{\hat{p}}, \hat{\sigma}^2_{\hat{\eta}}$ and $\hat{\eta}_t$. To estimate the coefficients $\hat{\phi}$ we can decide to use the small sample bias correction methods outlined in the previous section. [5]
3. Residuals re-sampling: in this step the residuals are centred. We name their distribution $\Phi_{\hat{\eta}}$. To avoid excessive notation, we do not use superscripts, but it should be clear that if we use bias correction ($\hat{\phi}^{BC}$) then the residuals will correspond to these coefficients, with a distribution $\Phi_{\hat{\eta}^{BC}}$, and analogously for Roy-Fuller's correction. We draw a random sample from the residuals' distribution function $\Phi_{\hat{\eta}}$ for $t = T+1, ..., T+H$, H being the maximum forecasting horizon considered.
4. Recursively generate factor's forecasts by using the AR estimated coefficients (with or without correction of the bias), the re-sampled residuals $\hat{\eta}_{T+h}$, and the last values for the common factor $f_{T-p+1}, ..., f_T$.

$$\hat{f}_{T+h} = \hat{\phi}_1 \hat{f}_{T+h-1} + \hat{\phi}_2 \hat{f}_{T+h-2} + ... + \hat{\phi}_p \hat{f}_{T+h-p} + \hat{\eta}_{T+h}. \tag{10}$$

Notice that by using the last values of f we are conditioning on the "observed"[6] sample realization (following Pascual et al., 2001).

Steps 3 and 4 are carried out B times ($B = 500$ in our simulation study), and they render an empirical forecast distribution for each factor, Φ_f. We employ Efron percentiles to obtain prediction intervals for f_{T+h}, $h = 1, ..., H$. Therefore, for a nominal coverage of $(1-\alpha)$ and forecasting horizon h, the interval for factor f is given by $[\Phi^{-1}_{\hat{f}_{T+h}}(\alpha/2), \Phi^{-1}_{\hat{f}_{T+h}}(1-\alpha/2)]$. As an advantage, this bootstrap approach does not assume normality in the errors of the models for the factors (Clements and Taylor, 2001; Fresoli et al., 2015, for an assessment of the effect of this assumption in forecasting densities of VAR(2) models with χ^2 errors).

5. Calculate forecasts for the series using the forecasts for each factor and the estimated weights (equation 11, in vector notation). We also obtain prediction intervals for the series employing Efron percentiles.

$$\hat{y}_{T+h} = \hat{\Omega} \hat{f}_{T+h} + \varepsilon_{T+h}, \tag{11}$$

where $\hat{y}_{T+h}$ is a vector that contains the forecasts for the N series, $\hat{\Omega}$ is the $(N \times r)$ estimated matrix of loadings, and $\hat{f}_{T+h}$ the forecasted factors $(r \times 1)$.

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4 See Alonso et al. (2004) for a discussion on how to introduce model selection in the bootstrap algorithm, and an assessment of results of alternative methods for the estimation of prediction intervals.

5 If so, we adapted package BootPR in the software R to use CSS to get the bias corrected $\hat{\phi}$.

6 "observed" is between quotes because the factors are not actually observed, but they themselves are estimates.

3. Experimental Design

We employ simulated datasets to illustrate the performance of the methodology. The data matrix Y has dimension $N \times T$, $N = 25$ being the number of time series included, and $T = 50, 100, 200$, the time dimension. Notice that T assumes three alternative values in order to allow comparison of the estimation's performance for different sample sizes. As Clements and Kim (2007) indicate, we expect the bias in the estimates of the AR coefficients to be worse the smaller the series' length.

The data matrix Y results from pre-specified AR common factors, orthonormal weight vectors (following Stock and Watson, 2002), and normally distributed idiosyncratic errors $\varepsilon_t \sim N(0, 0.1)$ with mean and standard deviation values like Alonso et al. (2011). In particular, we start with weights following model 2 of Alonso et al. (2011)[7] and then transform them into orthonormal vectors to obtain the following weights:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}^T = \begin{pmatrix} 0.19 & 0.19 & 0.19 & 0.19 & 0.38 & 0.19 & 0.19 & 0.19 & 0.19 & -0.1 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 & 0.19 \\ 0 & 0.02 & 0.02 & 0.02 & -0.31 & 0.02 & 0.02 & 0.02 & 0.02 & 0.51 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.72 \end{pmatrix}^T$$

Even if Principal Components allows to identify the factor space by assuming orthonormal common factors, we need to impose the condition that the first load of the second factor equals 0 in order to identify the individual common factors in a two-factor model, and model each of them as independent AR(p) processes. We experimented with alternative, in particular more diverse weights, and obtained similar results.

Additionally, the factors are created with no cross correlation between them, and the variance pertaining to the first factor is strictly greater than the one of the second factor $\sigma^2_{F1} > \sigma^2_{F2}$. The persistence in the factors will be such that their variances will maintain this relation. This is in line with assumption F1.b of Stock and Watson (2002) : "$E(F_t F_t') = \Sigma_{FF}$, where Σ_{FF} is a diagonal matrix with elements $\sigma_{ii} > \sigma_{jj} > 0$ for $i < j$".

We consider AR models of orders one and two for the common factors. The data generating process is deliberately simple in order to easily compare the effects of sample size, as well as the proximity to unit root.

Also, $\eta_{1,t} \sim N(0, 1)$ while $\eta_{2,t} \sim N(0, 0.5)$, according to the principle that the variance of the second factor should be smaller than the variance of the first factor.

We perform a Monte Carlo study (10,000 trials) to evaluate the benefits of small sample bias correction. The estimations will be evaluated in these two situations:

- the number of factors and their AR order are known
- the number of factors and their AR order are unknown

Even though known number of factors and AR order is an infeasible scenario, it will allow us to isolate any influence the estimation of these parameters may cast. The forecasting horizon ranges from 1 to 10. The bootstrap for the AR coefficients (in the bootstrap bias-corrected approach), and for the prediction intervals (all approaches) are based on 500 replications.

We compare the performance of not using a bias correction (denoted as *none* in the tables), the bootstrap bias-corrected estimator of Clements and Kim (2007) (denoted as *BC*), and Roy-Fuller estimates of Roy and Fuller (2001) (denoted as *RF*). We evaluate their performance for a 95% nominal coverage.

In order to assess performance, for the prediction intervals we obtain average coverage rates (C_m), average length (L_m), and CQ_m (a measure combining C_m and L_m) introduced in Alonso et al. (2002). The coverage rates are estimated as the average of the Monte Carlo trials coverage rates for the prediction intervals. The effective coverage rate in each Monte Carlo trial is the relative

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7 Alonso et al. (2011) loading matrix Ω is made up of vectors $\omega_1 = [1,1,1,1,2,1,1,1,1,-0.5,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]^T$, and $\omega_2 = 0.3 \times [0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,3]^T$.

frequency indicating the proportion of "true observations" included in the bootstrap interval. These "true processes" or continuations are created following Alonso et al. (2002). Furthermore, like these authors, we calculate a "theoretical" interval length (L_t) that can be used for comparison. Last, CQ_m is calculated as $CQ_m = |1 - C_m/C_t| + |1 - L_m/L_t|$, where C_t is the nominal coverage, and L_t the estimated theoretical mean interval length (Alonso et al., 2002).

4. Results for the Simulation

In this section we present the results for the Monte Carlo simulation. To make a clear presentation, we divide them in two parts. In the first part we present the results when the number of factors, R, and the factors' AR order, p, are known. In the second part (Section 4.2) we present the results when R and p are unknown and selected using IC_3 and BIC, respectively.

4.1. Number of Factors and AR orders Known

Firstly, we present the results for factors that follow AR(1) models. In order to ensure a higher variance of the first factor, its AR coefficient $\phi_{F1} = 0.975$ is greater than the corresponding one to the second factor, $\phi_{F2} = 0.90$, and the same for the variance of the noise ($\eta_{t,F1} = 1$ while $\eta_{t,F2} = 0.50$).

Results are obtained for sample sizes $T = 50, 100, 200$. Tables 1, 2, and 3 present the outcomes that correspond to five representative series y_t out of the $N = 25$ observed series generated. The tables in appendix A present results and explanations in detail for the factors.

In Table 1, for $T = 50$, we obtain that the coverage of the intervals, though usually well below the 95% theoretical value, is improved when using *BC* and *RF* (in comparison to *none*). Furthermore, the improvement is more noticeable the longer the forecasting horizon, i.e. $h = 10$ presents a greater improvement than $h = 1$. In this line, Clements and Taylor (2001) explain that the bias can increase with the forecasting horizon h because we power up the biased estimates to produce forecasts.

Be aware that that there are very small differences in the standard errors "*se*", presented between parenthesis. The average length of all the intervals, L_m, is larger when a correction is performed. Most often, the interval length for *none* underestimates the theoretical length, while bias correction renders intervals with length closer to the theoretical length reported. Last, CQ_m is never worse for the estimations with correction, with the exception of $Y25$ for $h = 1$. Recall that a value of $CQ_m = 0$ would mean a perfect estimation in the sense that both coverage and length coincide with the theoretical values.

Table 2 corresponds to a sample size $T = 100$. C_m of *BC* and *RF* are always better than that for *none*. And again, the improvement of using corrections is more noticeable in long ($h = 10$) than short horizons ($h = 1$). The gains in C_m of using *BC* or *RF* are smaller than those for the smaller sample size of $T = 50$, which is consistent with the idea that, the smaller the sample, the greater the bias, and the more useful the role of bias correction in the AR estimates. L_m continues to be greater when a correction is performed, and in most cases closer to L_t. Furthermore, in some cases, for the shortest horizons ($h = 1$) the value of CQ_m for *none* results equal than that for *BC* or *RF*.

Table 3 for $T = 200$ obtains that coverage C_m is always better for *BC* and *RF* than for *none*, and like in the previous cases, the improvement is more noticeable for long than for short horizons. As expected, the improvement in terms of coverage tends to be smaller (across forecasting horizons and estimation techniques) than for the smaller sample sizes. Like in the previous cases, L_m tends to be greater (and closer to L_t) when some type of correction is performed. CQ_m tends to be better (equal for $h = 1$) when performing a correction, though the improvements are usually not as good as those for smaller sample sizes. Finally, as expected, the behavior (in terms of C_m, L_m, and CQ_m) of the three procedures improves with the series' length.

Tables 4 to 6 provide with the results when the factors follow AR(2) models instead. The two roots for the characteristic equation of the factors are: $a^1_{F1} = 0.50, a^2_{F1} = 0.975, a^1_{F2} = 0.50, a^2_{F2} = 0.90$. The findings are similar to those obtained for AR(1) models. As before, the improvements from using small sample bias corrections deteriorate as the sample size increases from $T = 50$ to $T = 200$. Again, the improvements from the corrections are more noticeable as the prediction horizon increases. And even though coverage is always better for the estimations with bias correction, the interval length L_m, and CQ_m are sometimes similar for corrected and *none*, specially for $h = 1$.

Table 1. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 50$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	90.68 (0.052)	0.98 (0.001)	1.05 (0.000)	0.11
		BC	91.11 (0.049)	0.99 (0.001)	1.05 (0.000)	0.10
		RF	91.30 (0.048)	0.99 (0.001)	1.05 (0.000)	0.09
	h=5	none	85.29 (0.088)	1.67 (0.003)	2.01 (0.001)	0.27
		BC	88.91 (0.077)	1.87 (0.004)	2.01 (0.001)	0.13
		RF	90.25 (0.072)	1.92 (0.004)	2.01 (0.001)	0.10
	h=10	none	80.36 (0.112)	1.92 (0.005)	2.55 (0.001)	0.40
		BC	86.33 (0.105)	2.34 (0.006)	2.55 (0.001)	0.17
		RF	89.11 (0.094)	2.47 (0.007)	2.55 (0.001)	0.09
Y2	h=1	none	87.36 (0.059)	0.71 (0.001)	0.84 (0.000)	0.24
		BC	87.92 (0.057)	0.72 (0.001)	0.84 (0.000)	0.22
		RF	88.17 (0.055)	0.72 (0.001)	0.84 (0.000)	0.22
	h=5	none	83.73 (0.090)	1.30 (0.002)	1.63 (0.001)	0.33
		BC	87.78 (0.080)	1.45 (0.003)	1.63 (0.001)	0.19
		RF	89.54 (0.074)	1.48 (0.003)	1.63 (0.001)	0.15
	h=10	none	77.99 (0.121)	1.54 (0.004)	2.15 (0.001)	0.46
		BC	84.86 (0.114)	1.89 (0.005)	2.15 (0.001)	0.23
		RF	88.19 (0.105)	1.98 (0.005)	2.15 (0.001)	0.15
Y5	h=1	none	91.15 (0.048)	1.55 (0.002)	1.65 (0.001)	0.10
		BC	91.58 (0.045)	1.57 (0.002)	1.65 (0.001)	0.09
		RF	91.84 (0.044)	1.57 (0.002)	1.65 (0.001)	0.08
	h=5	none	84.93 (0.088)	2.77 (0.005)	3.38 (0.001)	0.29
		BC	88.79 (0.076)	3.10 (0.006)	3.38 (0.001)	0.15
		RF	90.41 (0.070)	3.18 (0.006)	3.38 (0.001)	0.11
	h=10	none	79.16 (0.116)	3.26 (0.008)	4.43 (0.002)	0.43
		BC	85.71 (0.109)	4.00 (0.010)	4.43 (0.002)	0.19
		RF	88.90 (0.097)	4.19 (0.011)	4.43 (0.002)	0.12
Y10	h=1	none	92.13 (0.051)	1.13 (0.002)	1.13 (0.000)	0.03
		BC	92.48 (0.048)	1.14 (0.002)	1.13 (0.000)	0.03
		RF	92.56 (0.048)	1.14 (0.002)	1.13 (0.000)	0.03
	h=5	none	86.74 (0.085)	1.78 (0.004)	2.05 (0.001)	0.22
		BC	89.83 (0.077)	1.99 (0.005)	2.05 (0.001)	0.08
		RF	90.67 (0.075)	2.05 (0.005)	2.05 (0.001)	0.05
	h=10	none	83.36 (0.101)	1.95 (0.005)	2.43 (0.001)	0.32
		BC	87.87 (0.099)	2.36 (0.007)	2.43 (0.001)	0.10
		RF	89.64 (0.092)	2.50 (0.008)	2.43 (0.001)	0.09
Y25	h=1	none	92.78 (0.048)	1.66 (0.003)	1.64 (0.001)	0.04
		BC	93.14 (0.045)	1.68 (0.003)	1.64 (0.001)	0.04
		RF	93.25 (0.044)	1.68 (0.003)	1.64 (0.001)	0.05
	h=5	none	86.66 (0.085)	2.66 (0.005)	3.07 (0.001)	0.22
		BC	89.83 (0.077)	2.97 (0.007)	3.07 (0.001)	0.09
		RF	90.82 (0.073)	3.06 (0.007)	3.07 (0.001)	0.05
	h=10	none	82.85 (0.104)	2.94 (0.007)	3.71 (0.002)	0.33
		BC	87.62 (0.101)	3.56 (0.010)	3.71 (0.002)	0.12
		RF	89.66 (0.090)	3.77 (0.011)	3.71 (0.002)	0.07

Table 2. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 100$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	91.93 (0.037)	0.99 (0.001)	1.05 (0.000)	0.09
		BC	92.06 (0.036)	0.99 (0.001)	1.05 (0.000)	0.09
		RF	92.13 (0.036)	1.00 (0.001)	1.05 (0.000)	0.08
	h=5	none	89.84 (0.057)	1.82 (0.003)	2.01 (0.001)	0.15
		BC	91.77 (0.049)	1.94 (0.003)	2.01 (0.001)	0.07
		RF	92.26 (0.047)	1.97 (0.003)	2.01 (0.001)	0.05
	h=10	none	87.01 (0.076)	2.16 (0.004)	2.55 (0.001)	0.23
		BC	90.53 (0.067)	2.45 (0.005)	2.55 (0.001)	0.08
		RF	91.55 (0.061)	2.52 (0.005)	2.55 (0.001)	0.05
Y2	h=1	none	89.50 (0.040)	0.73 (0.001)	0.84 (0.000)	0.19
		BC	89.64 (0.039)	0.73 (0.001)	0.84 (0.000)	0.18
		RF	89.82 (0.039)	0.74 (0.001)	0.84 (0.000)	0.18
	h=5	none	89.08 (0.057)	1.43 (0.002)	1.63 (0.001)	0.19
		BC	91.21 (0.049)	1.53 (0.002)	1.63 (0.001)	0.10
		RF	91.95 (0.045)	1.56 (0.002)	1.63 (0.001)	0.08
	h=10	none	85.86 (0.082)	1.77 (0.003)	2.15 (0.001)	0.27
		BC	89.92 (0.072)	2.03 (0.004)	2.15 (0.001)	0.11
		RF	91.25 (0.066)	2.09 (0.004)	2.15 (0.001)	0.07
Y5	h=1	none	92.66 (0.033)	1.59 (0.002)	1.65 (0.001)	0.06
		BC	92.79 (0.032)	1.59 (0.002)	1.65 (0.001)	0.06
		RF	92.93 (0.031)	1.60 (0.002)	1.65 (0.001)	0.05
	h=5	none	89.89 (0.056)	3.04 (0.004)	3.38 (0.001)	0.15
		BC	91.89 (0.048)	3.26 (0.004)	3.38 (0.001)	0.07
		RF	92.54 (0.045)	3.30 (0.004)	3.38 (0.001)	0.05
	h=10	none	86.55 (0.080)	3.72 (0.007)	4.43 (0.002)	0.25
		BC	90.41 (0.069)	4.25 (0.008)	4.43 (0.002)	0.09
		RF	91.64 (0.064)	4.37 (0.008)	4.43 (0.002)	0.05
Y10	h=1	none	93.00 (0.034)	1.12 (0.001)	1.14 (0.000)	0.03
		BC	93.11 (0.034)	1.12 (0.001)	1.14 (0.000)	0.03
		RF	93.11 (0.034)	1.12 (0.001)	1.14 (0.000)	0.03
	h=5	none	90.74 (0.054)	1.91 (0.003)	2.05 (0.001)	0.12
		BC	92.40 (0.049)	2.03 (0.003)	2.05 (0.001)	0.04
		RF	92.54 (0.049)	2.05 (0.003)	2.05 (0.001)	0.03
	h=10	none	88.67 (0.067)	2.14 (0.004)	2.42 (0.001)	0.18
		BC	91.33 (0.063)	2.40 (0.005)	2.42 (0.001)	0.05
		RF	91.76 (0.062)	2.45 (0.005)	2.42 (0.001)	0.05
Y25	h=1	none	93.73 (0.032)	1.66 (0.002)	1.64 (0.001)	0.03
		BC	93.85 (0.031)	1.66 (0.002)	1.64 (0.001)	0.03
		RF	93.87 (0.031)	1.67 (0.002)	1.64 (0.001)	0.03
	h=5	none	90.84 (0.054)	2.86 (0.004)	3.07 (0.001)	0.11
		BC	92.54 (0.048)	3.05 (0.005)	3.07 (0.001)	0.03
		RF	92.76 (0.047)	3.09 (0.005)	3.07 (0.001)	0.03
	h=10	none	88.49 (0.069)	3.26 (0.006)	3.71 (0.002)	0.19
		BC	91.35 (0.063)	3.67 (0.007)	3.71 (0.002)	0.05
		RF	91.94 (0.060)	3.76 (0.008)	3.71 (0.002)	0.05

Table 3. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 200$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	92.55 (0.027)	1.00 (0.001)	1.05 (0.000)	0.08
		BC	92.59 (0.027)	1.00 (0.001)	1.05 (0.000)	0.08
		RF	92.59 (0.027)	1.00 (0.001)	1.05 (0.000)	0.08
	h=5	none	92.36 (0.035)	1.91 (0.002)	2.01 (0.001)	0.08
		BC	93.29 (0.032)	1.98 (0.002)	2.01 (0.001)	0.03
		RF	93.42 (0.031)	1.99 (0.002)	2.01 (0.001)	0.03
	h=10	none	90.96 (0.047)	2.33 (0.003)	2.55 (0.001)	0.13
		BC	92.81 (0.041)	2.51 (0.003)	2.55 (0.001)	0.04
		RF	93.11 (0.040)	2.55 (0.003)	2.55 (0.001)	0.02
Y2	h=1	none	90.66 (0.029)	0.74 (0.001)	0.84 (0.000)	0.16
		BC	90.69 (0.029)	0.74 (0.001)	0.84 (0.000)	0.16
		RF	90.71 (0.029)	0.75 (0.001)	0.84 (0.000)	0.16
	h=5	none	91.86 (0.034)	1.51 (0.001)	1.63 (0.001)	0.10
		BC	92.93 (0.030)	1.58 (0.001)	1.63 (0.001)	0.06
		RF	93.10 (0.029)	1.59 (0.001)	1.63 (0.001)	0.05
	h=10	none	90.44 (0.049)	1.94 (0.002)	2.15 (0.001)	0.15
		BC	92.53 (0.043)	2.10 (0.003)	2.15 (0.001)	0.05
		RF	92.91 (0.041)	2.14 (0.003)	2.15 (0.001)	0.02
Y5	h=1	none	93.47 (0.024)	1.61 (0.001)	1.65 (0.001)	0.04
		BC	93.52 (0.024)	1.61 (0.001)	1.65 (0.001)	0.04
		RF	93.51 (0.024)	1.61 (0.001)	1.65 (0.001)	0.04
	h=5	none	92.52 (0.034)	3.21 (0.003)	3.37 (0.001)	0.07
		BC	93.50 (0.030)	3.34 (0.003)	3.37 (0.001)	0.03
		RF	93.68 (0.029)	3.37 (0.003)	3.37 (0.001)	0.02
	h=10	none	90.93 (0.048)	4.05 (0.005)	4.42 (0.002)	0.13
		BC	92.89 (0.042)	4.39 (0.006)	4.42 (0.002)	0.03
		RF	93.24 (0.040)	4.47 (0.006)	4.42 (0.002)	0.03
Y10	h=1	none	93.33 (0.026)	1.11 (0.001)	1.14 (0.000)	0.04
		BC	93.36 (0.026)	1.11 (0.001)	1.14 (0.000)	0.04
		RF	93.35 (0.026)	1.11 (0.001)	1.14 (0.000)	0.04
	h=5	none	92.84 (0.034)	1.98 (0.002)	2.05 (0.001)	0.06
		BC	93.66 (0.031)	2.04 (0.002)	2.05 (0.001)	0.02
		RF	93.68 (0.031)	2.05 (0.002)	2.05 (0.001)	0.01
	h=10	none	91.68 (0.044)	2.27 (0.003)	2.42 (0.001)	0.10
		BC	93.11 (0.040)	2.41 (0.003)	2.42 (0.001)	0.03
		RF	93.21 (0.040)	2.43 (0.003)	2.42 (0.001)	0.02
Y25	h=1	none	94.09 (0.024)	1.65 (0.001)	1.64 (0.001)	0.02
		BC	94.11 (0.024)	1.65 (0.001)	1.64 (0.001)	0.02
		RF	94.11 (0.024)	1.65 (0.001)	1.64 (0.001)	0.02
	h=5	none	93.00 (0.034)	2.98 (0.003)	3.07 (0.001)	0.05
		BC	93.83 (0.031)	3.09 (0.003)	3.07 (0.001)	0.02
		RF	93.84 (0.031)	3.10 (0.003)	3.07 (0.001)	0.02
	h=10	none	91.72 (0.044)	3.48 (0.005)	3.70 (0.002)	0.10
		BC	93.26 (0.039)	3.71 (0.005)	3.70 (0.002)	0.02
		RF	93.37 (0.039)	3.74 (0.005)	3.70 (0.002)	0.03

Table 4. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 50$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	89.88 (0.059)	0.98 (0.001)	1.05 (0.000)	0.12
		BC	90.46 (0.056)	1.00 (0.001)	1.05 (0.000)	0.10
		RF	90.72 (0.054)	1.00 (0.002)	1.05 (0.000)	0.10
	h=5	none	83.06 (0.116)	2.75 (0.006)	3.29 (0.001)	0.29
		BC	87.28 (0.101)	3.06 (0.007)	3.29 (0.001)	0.15
		RF	88.46 (0.089)	3.06 (0.007)	3.29 (0.001)	0.14
	h=10	none	76.52 (0.144)	3.39 (0.010)	4.61 (0.002)	0.46
		BC	83.76 (0.137)	4.22 (0.013)	4.61 (0.002)	0.20
		RF	86.67 (0.113)	4.29 (0.013)	4.61 (0.002)	0.16
Y2	h=1	none	86.06 (0.065)	0.70 (0.001)	0.84 (0.000)	0.26
		BC	86.71 (0.062)	0.71 (0.001)	0.84 (0.000)	0.25
		RF	87.10 (0.061)	0.71 (0.001)	0.84 (0.000)	0.24
	h=5	none	81.90 (0.117)	2.13 (0.005)	2.65 (0.001)	0.33
		BC	86.22 (0.105)	2.37 (0.006)	2.65 (0.001)	0.20
		RF	87.96 (0.090)	2.37 (0.005)	2.65 (0.001)	0.18
	h=10	none	74.35 (0.152)	2.75 (0.008)	3.89 (0.002)	0.51
		BC	82.22 (0.146)	3.41 (0.011)	3.89 (0.002)	0.26
		RF	85.96 (0.120)	3.47 (0.010)	3.89 (0.002)	0.20
Y5	h=1	none	90.06 (0.055)	1.53 (0.002)	1.65 (0.001)	0.12
		BC	90.66 (0.052)	1.55 (0.002)	1.65 (0.001)	0.10
		RF	91.00 (0.050)	1.56 (0.002)	1.65 (0.001)	0.10
	h=5	none	82.51 (0.116)	4.55 (0.010)	5.55 (0.002)	0.31
		BC	86.83 (0.101)	5.07 (0.012)	5.55 (0.002)	0.17
		RF	88.41 (0.087)	5.06 (0.011)	5.55 (0.002)	0.16
	h=10	none	75.10 (0.148)	5.79 (0.018)	8.08 (0.003)	0.49
		BC	82.87 (0.141)	7.20 (0.023)	8.08 (0.003)	0.24
		RF	86.36 (0.115)	7.32 (0.022)	8.08 (0.003)	0.18
Y10	h=1	none	91.51 (0.061)	1.15 (0.002)	1.13 (0.000)	0.05
		BC	92.00 (0.056)	1.16 (0.002)	1.13 (0.000)	0.06
		RF	92.20 (0.055)	1.17 (0.002)	1.13 (0.000)	0.06
	h=5	none	84.21 (0.117)	2.94 (0.007)	3.36 (0.001)	0.24
		BC	88.10 (0.103)	3.28 (0.008)	3.36 (0.001)	0.10
		RF	88.78 (0.093)	3.28 (0.008)	3.36 (0.001)	0.09
	h=10	none	79.46 (0.136)	3.42 (0.010)	4.37 (0.002)	0.38
		BC	85.39 (0.130)	4.24 (0.014)	4.37 (0.002)	0.13
		RF	87.24 (0.110)	4.33 (0.014)	4.37 (0.002)	0.09
Y25	h=1	none	92.14 (0.057)	1.69 (0.003)	1.64 (0.001)	0.07
		BC	92.66 (0.053)	1.71 (0.003)	1.64 (0.001)	0.07
		RF	92.85 (0.051)	1.72 (0.003)	1.64 (0.001)	0.07
	h=5	none	84.19 (0.115)	4.41 (0.011)	5.06 (0.002)	0.24
		BC	88.16 (0.100)	4.92 (0.012)	5.06 (0.002)	0.10
		RF	88.94 (0.091)	4.92 (0.012)	5.06 (0.002)	0.09
	h=10	none	78.91 (0.136)	5.21 (0.015)	6.73 (0.003)	0.40
		BC	85.20 (0.128)	6.46 (0.021)	6.73 (0.003)	0.14
		RF	87.36 (0.109)	6.59 (0.021)	6.73 (0.003)	0.10

Table 5. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 100$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	91.70 (0.041)	1.00 (0.001)	1.05 (0.000)	0.08
		BC	91.91 (0.039)	1.01 (0.001)	1.05 (0.000)	0.08
		RF	91.97 (0.039)	1.01 (0.001)	1.05 (0.000)	0.07
	h=5	none	89.55 (0.067)	3.02 (0.005)	3.29 (0.001)	0.14
		BC	91.39 (0.057)	3.20 (0.005)	3.29 (0.001)	0.07
		RF	91.52 (0.054)	3.19 (0.005)	3.29 (0.001)	0.07
	h=10	none	85.91 (0.092)	3.92 (0.008)	4.61 (0.002)	0.24
		BC	89.71 (0.079)	4.45 (0.010)	4.61 (0.002)	0.09
		RF	90.43 (0.071)	4.49 (0.010)	4.61 (0.002)	0.07
Y2	h=1	none	89.11 (0.043)	0.73 (0.001)	0.84 (0.000)	0.19
		BC	89.39 (0.042)	0.74 (0.001)	0.84 (0.000)	0.18
		RF	89.50 (0.041)	0.74 (0.001)	0.84 (0.000)	0.18
	h=5	none	89.12 (0.066)	2.38 (0.004)	2.65 (0.001)	0.16
		BC	91.00 (0.057)	2.52 (0.004)	2.65 (0.001)	0.09
		RF	91.42 (0.051)	2.51 (0.004)	2.65 (0.001)	0.09
	h=10	none	85.03 (0.096)	3.25 (0.007)	3.89 (0.002)	0.27
		BC	89.14 (0.084)	3.69 (0.008)	3.89 (0.002)	0.11
		RF	90.25 (0.073)	3.71 (0.008)	3.89 (0.002)	0.09
Y5	h=1	none	92.31 (0.037)	1.59 (0.002)	1.65 (0.001)	0.06
		BC	92.52 (0.035)	1.60 (0.002)	1.65 (0.001)	0.06
		RF	92.64 (0.035)	1.60 (0.002)	1.65 (0.001)	0.05
	h=5	none	89.42 (0.067)	5.06 (0.008)	5.55 (0.002)	0.15
		BC	91.29 (0.057)	5.36 (0.008)	5.55 (0.002)	0.07
		RF	91.62 (0.052)	5.34 (0.008)	5.55 (0.002)	0.07
	h=10	none	85.39 (0.095)	6.80 (0.014)	8.08 (0.003)	0.26
		BC	89.40 (0.082)	7.73 (0.016)	8.08 (0.003)	0.10
		RF	90.39 (0.072)	7.78 (0.016)	8.08 (0.003)	0.09
Y10	h=1	none	92.84 (0.040)	1.14 (0.001)	1.13 (0.000)	0.03
		BC	93.01 (0.039)	1.15 (0.001)	1.13 (0.000)	0.03
		RF	93.00 (0.039)	1.15 (0.001)	1.13 (0.000)	0.03
	h=5	none	90.18 (0.066)	3.16 (0.005)	3.35 (0.001)	0.11
		BC	91.84 (0.058)	3.34 (0.006)	3.35 (0.001)	0.04
		RF	91.69 (0.057)	3.33 (0.006)	3.35 (0.001)	0.04
	h=10	none	87.20 (0.085)	3.82 (0.008)	4.37 (0.002)	0.21
		BC	90.37 (0.077)	4.33 (0.010)	4.37 (0.002)	0.06
		RF	90.48 (0.073)	4.37 (0.010)	4.37 (0.002)	0.05
Y25	h=1	none	93.53 (0.037)	1.69 (0.002)	1.64 (0.001)	0.05
		BC	93.66 (0.036)	1.70 (0.002)	1.64 (0.001)	0.05
		RF	93.67 (0.036)	1.70 (0.002)	1.64 (0.001)	0.05
	h=5	none	90.21 (0.065)	4.75 (0.008)	5.05 (0.002)	0.11
		BC	91.89 (0.057)	5.04 (0.008)	5.05 (0.002)	0.04
		RF	91.82 (0.055)	5.02 (0.008)	5.05 (0.002)	0.04
	h=10	none	87.03 (0.085)	5.86 (0.012)	6.72 (0.003)	0.21
		BC	90.29 (0.076)	6.63 (0.014)	6.72 (0.003)	0.06
		RF	90.59 (0.071)	6.70 (0.014)	6.72 (0.003)	0.05

Table 6. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 200$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	92.51 (0.029)	1.01 (0.001)	1.05 (0.000)	0.07
		BC	92.55 (0.029)	1.01 (0.001)	1.05 (0.000)	0.07
		RF	92.58 (0.029)	1.01 (0.001)	1.05 (0.000)	0.07
	h=5	none	92.36 (0.039)	3.15 (0.004)	3.29 (0.001)	0.07
		BC	93.14 (0.035)	3.25 (0.004)	3.29 (0.001)	0.03
		RF	93.05 (0.035)	3.24 (0.004)	3.29 (0.001)	0.03
	h=10	none	90.67 (0.053)	4.25 (0.006)	4.61 (0.002)	0.12
		BC	92.47 (0.046)	4.57 (0.007)	4.61 (0.002)	0.04
		RF	92.48 (0.045)	4.58 (0.007)	4.61 (0.002)	0.03
Y2	h=1	none	90.50 (0.030)	0.75 (0.001)	0.84 (0.000)	0.16
		BC	90.57 (0.030)	0.75 (0.001)	0.84 (0.000)	0.16
		RF	90.65 (0.030)	0.75 (0.001)	0.84 (0.000)	0.15
	h=5	none	92.16 (0.038)	2.51 (0.003)	2.65 (0.001)	0.08
		BC	92.95 (0.035)	2.59 (0.003)	2.65 (0.001)	0.04
		RF	92.94 (0.033)	2.59 (0.003)	2.65 (0.001)	0.04
	h=10	none	90.38 (0.055)	3.55 (0.005)	3.89 (0.002)	0.13
		BC	92.23 (0.049)	3.83 (0.005)	3.89 (0.002)	0.04
		RF	92.37 (0.046)	3.84 (0.005)	3.89 (0.002)	0.04
Y5	h=1	none	93.35 (0.026)	1.62 (0.001)	1.65 (0.001)	0.04
		BC	93.41 (0.026)	1.62 (0.001)	1.65 (0.001)	0.03
		RF	93.47 (0.025)	1.62 (0.001)	1.65 (0.001)	0.03
	h=5	none	92.42 (0.038)	5.33 (0.006)	5.56 (0.002)	0.07
		BC	93.19 (0.034)	5.49 (0.006)	5.56 (0.002)	0.03
		RF	93.14 (0.033)	5.48 (0.006)	5.56 (0.002)	0.03
	h=10	none	90.58 (0.054)	7.43 (0.010)	8.08 (0.003)	0.13
		BC	92.41 (0.048)	8.00 (0.011)	8.08 (0.003)	0.04
		RF	92.53 (0.045)	8.03 (0.011)	8.08 (0.003)	0.03
Y10	h=1	none	93.24 (0.029)	1.13 (0.001)	1.13 (0.000)	0.02
		BC	93.28 (0.029)	1.13 (0.001)	1.13 (0.000)	0.02
		RF	93.27 (0.029)	1.13 (0.001)	1.13 (0.000)	0.02
	h=5	none	92.56 (0.038)	3.24 (0.004)	3.35 (0.001)	0.06
		BC	93.29 (0.035)	3.34 (0.004)	3.35 (0.001)	0.02
		RF	93.11 (0.036)	3.33 (0.004)	3.35 (0.001)	0.03
	h=10	none	91.04 (0.050)	4.06 (0.006)	4.37 (0.002)	0.11
		BC	92.65 (0.045)	4.34 (0.006)	4.37 (0.002)	0.03
		RF	92.49 (0.047)	4.34 (0.006)	4.37 (0.002)	0.03
Y25	h=1	none	93.91 (0.027)	1.67 (0.001)	1.64 (0.001)	0.03
		BC	93.94 (0.027)	1.67 (0.001)	1.64 (0.001)	0.03
		RF	93.94 (0.028)	1.68 (0.001)	1.64 (0.001)	0.04
	h=5	none	92.57 (0.037)	4.89 (0.006)	5.06 (0.002)	0.06
		BC	93.29 (0.034)	5.03 (0.006)	5.06 (0.002)	0.02
		RF	93.17 (0.035)	5.02 (0.006)	5.06 (0.002)	0.03
	h=10	none	90.96 (0.050)	6.23 (0.009)	6.73 (0.003)	0.12
		BC	92.54 (0.045)	6.66 (0.010)	6.73 (0.003)	0.04
		RF	92.46 (0.046)	6.67 (0.010)	6.73 (0.003)	0.04

4.2. Number of Factors and AR orders Unknown

We consider a model of two factors in this experiment, and use a sample size of $T = 100$ (see Appendix C for other values of T). The number of factors is estimated by IC_3 of Bai and Ng (2002), as explained in Sub-Section 2.1. This criterion correctly estimated the number of factors R in more than 99.9% of cases.

We consider factors that are AR(2), as Clements and Kim (2007) explain, in order to allow under and over specification of p. The lag order estimated is restricted to at most six, and for selection criteria we compare AICc and BIC. We did not endogenise the selection of p in the bootstrap algorithm because of the small improvements obtained by doing so in Clements and Kim (2007).

Table 7 presents the results when we use BIC as the criteria for selecting p, and Table 8 presents the results when AICc is the criteria for selecting p.

In both cases, for the selected series coverage C_m, length L_m, and CQ tend to be better for the models that use bias-corrected estimators than for *none* (the correction never results in worse off results than *none*). Furthermore, we can verify the same pattern than in the previous section: improvements become more noticeable the longer the forecasting horizon.

Last, comparing the two selection criteria we can see that BIC does a much better job selecting p than AICc (see Table 9 for a comparison of the distribution of $\hat{p}$), and it translates in better values of C_m as well as a slight general improvement in CQ_m.

Table 7. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created with both common factors following AR(2) processes with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 100$. Nominal coverage 95%. IC_3 used in the estimation of R, BIC used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	91.39 (0.042)	1.00 (0.001)	1.05 (0.000)	0.09
		BC	91.55 (0.040)	1.00 (0.001)	1.05 (0.000)	0.09
		RF	91.66 (0.040)	1.00 (0.001)	1.05 (0.000)	0.08
	h=5	none	89.27 (0.068)	3.01 (0.005)	3.29 (0.001)	0.15
		BC	91.07 (0.059)	3.18 (0.005)	3.29 (0.001)	0.07
		RF	91.24 (0.056)	3.17 (0.005)	3.29 (0.001)	0.07
	h=10	none	85.66 (0.094)	3.92 (0.008)	4.61 (0.002)	0.25
		BC	89.43 (0.082)	4.45 (0.010)	4.61 (0.002)	0.09
		RF	90.16 (0.073)	4.48 (0.010)	4.61 (0.002)	0.08
Y2	h=1	none	88.72 (0.045)	0.73 (0.001)	0.84 (0.000)	0.20
		BC	88.95 (0.045)	0.73 (0.001)	0.84 (0.000)	0.20
		RF	89.10 (0.043)	0.73 (0.001)	0.84 (0.000)	0.19
	h=5	none	88.71 (0.071)	2.37 (0.004)	2.65 (0.001)	0.17
		BC	90.56 (0.063)	2.51 (0.004)	2.65 (0.001)	0.10
		RF	91.02 (0.056)	2.50 (0.004)	2.65 (0.001)	0.10
	h=10	none	84.70 (0.102)	3.25 (0.007)	3.88 (0.002)	0.27
		BC	88.75 (0.091)	3.69 (0.008)	3.88 (0.002)	0.12
		RF	89.92 (0.078)	3.71 (0.008)	3.88 (0.002)	0.10
Y5	h=1	none	91.98 (0.039)	1.58 (0.002)	1.65 (0.001)	0.07
		BC	92.18 (0.038)	1.59 (0.002)	1.65 (0.001)	0.07
		RF	92.32 (0.037)	1.59 (0.002)	1.65 (0.001)	0.06
	h=5	none	89.07 (0.069)	5.04 (0.008)	5.55 (0.002)	0.15
		BC	90.91 (0.060)	5.33 (0.009)	5.55 (0.002)	0.08
		RF	91.25 (0.055)	5.31 (0.008)	5.55 (0.002)	0.08
	h=10	none	85.10 (0.099)	6.80 (0.014)	8.07 (0.003)	0.26
		BC	89.05 (0.087)	7.72 (0.017)	8.07 (0.003)	0.11
		RF	90.09 (0.075)	7.76 (0.017)	8.07 (0.003)	0.09
Y10	h=1	none	92.56 (0.041)	1.14 (0.001)	1.14 (0.000)	0.03
		BC	92.70 (0.040)	1.14 (0.001)	1.14 (0.000)	0.03
		RF	92.73 (0.040)	1.14 (0.001)	1.14 (0.000)	0.03
	h=5	none	89.89 (0.068)	3.15 (0.005)	3.35 (0.001)	0.12
		BC	91.54 (0.061)	3.33 (0.006)	3.35 (0.001)	0.04
		RF	91.46 (0.060)	3.32 (0.006)	3.35 (0.001)	0.05
	h=10	none	86.98 (0.087)	3.83 (0.008)	4.37 (0.002)	0.21
		BC	90.09 (0.080)	4.33 (0.010)	4.37 (0.002)	0.06
		RF	90.24 (0.076)	4.36 (0.010)	4.37 (0.002)	0.05
Y25	h=1	none	93.26 (0.039)	1.68 (0.002)	1.64 (0.001)	0.04
		BC	93.41 (0.038)	1.69 (0.002)	1.64 (0.001)	0.05
		RF	93.45 (0.037)	1.69 (0.002)	1.64 (0.001)	0.05
	h=5	none	89.85 (0.068)	4.75 (0.008)	5.06 (0.002)	0.12
		BC	91.55 (0.060)	5.02 (0.009)	5.06 (0.002)	0.04
		RF	91.51 (0.058)	5.01 (0.008)	5.06 (0.002)	0.05
	h=10	none	86.69 (0.090)	5.88 (0.012)	6.72 (0.003)	0.21
		BC	90.04 (0.081)	6.65 (0.015)	6.72 (0.003)	0.06
		RF	90.35 (0.074)	6.68 (0.015)	6.72 (0.003)	0.05

Table 8. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 100$. Nominal coverage 95%. IC_3 used in the estimation of R, AICc used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	90.73 (0.045)	0.98 (0.001)	1.05 (0.000)	0.11
		BC	90.88 (0.044)	0.99 (0.001)	1.05 (0.000)	0.11
		RF	91.00 (0.044)	0.99 (0.001)	1.05 (0.000)	0.10
	h=5	none	88.35 (0.073)	2.97 (0.005)	3.29 (0.001)	0.17
		BC	90.27 (0.065)	3.14 (0.005)	3.29 (0.001)	0.09
		RF	90.42 (0.061)	3.12 (0.005)	3.29 (0.001)	0.10
	h=10	none	84.77 (0.100)	3.90 (0.008)	4.61 (0.002)	0.26
		BC	88.59 (0.089)	4.41 (0.010)	4.61 (0.002)	0.11
		RF	89.25 (0.079)	4.39 (0.010)	4.61 (0.002)	0.11
Y2	h=1	none	87.98 (0.048)	0.72 (0.001)	0.84 (0.000)	0.22
		BC	88.27 (0.048)	0.72 (0.001)	0.84 (0.000)	0.21
		RF	88.41 (0.047)	0.72 (0.001)	0.84 (0.000)	0.21
	h=5	none	87.75 (0.076)	2.34 (0.004)	2.65 (0.001)	0.19
		BC	89.69 (0.069)	2.47 (0.004)	2.65 (0.001)	0.12
		RF	90.15 (0.062)	2.46 (0.004)	2.65 (0.001)	0.12
	h=10	none	83.76 (0.109)	3.22 (0.007)	3.88 (0.002)	0.29
		BC	87.84 (0.099)	3.65 (0.008)	3.88 (0.002)	0.14
		RF	88.95 (0.084)	3.63 (0.008)	3.88 (0.002)	0.13
Y5	h=1	none	91.35 (0.042)	1.56 (0.002)	1.65 (0.001)	0.09
		BC	91.53 (0.041)	1.57 (0.002)	1.65 (0.001)	0.08
		RF	91.68 (0.040)	1.57 (0.002)	1.65 (0.001)	0.08
	h=5	none	88.13 (0.074)	4.96 (0.008)	5.55 (0.002)	0.18
		BC	90.09 (0.066)	5.26 (0.008)	5.55 (0.002)	0.10
		RF	90.46 (0.060)	5.23 (0.008)	5.55 (0.002)	0.11
	h=10	none	84.19 (0.106)	6.75 (0.015)	8.07 (0.003)	0.28
		BC	88.20 (0.095)	7.64 (0.017)	8.07 (0.003)	0.12
		RF	89.16 (0.081)	7.61 (0.017)	8.07 (0.003)	0.12
Y10	h=1	none	91.91 (0.047)	1.12 (0.001)	1.14 (0.000)	0.04
		BC	92.00 (0.046)	1.13 (0.001)	1.14 (0.000)	0.04
		RF	92.07 (0.046)	1.13 (0.001)	1.14 (0.000)	0.04
	h=5	none	89.02 (0.076)	3.12 (0.005)	3.35 (0.001)	0.13
		BC	90.76 (0.068)	3.29 (0.006)	3.35 (0.001)	0.06
		RF	90.60 (0.066)	3.27 (0.006)	3.35 (0.001)	0.07
	h=10	none	86.09 (0.094)	3.81 (0.008)	4.37 (0.002)	0.22
		BC	89.26 (0.088)	4.28 (0.010)	4.37 (0.002)	0.08
		RF	89.27 (0.082)	4.27 (0.010)	4.37 (0.002)	0.08
Y25	h=1	none	92.55 (0.044)	1.66 (0.002)	1.64 (0.001)	0.04
		BC	92.75 (0.043)	1.66 (0.002)	1.64 (0.001)	0.04
		RF	92.80 (0.043)	1.67 (0.002)	1.64 (0.001)	0.04
	h=5	none	88.98 (0.075)	4.69 (0.008)	5.06 (0.002)	0.14
		BC	90.78 (0.066)	4.96 (0.009)	5.06 (0.002)	0.06
		RF	90.70 (0.064)	4.93 (0.008)	5.06 (0.002)	0.07
	h=10	none	85.80 (0.097)	5.84 (0.013)	6.72 (0.003)	0.23
		BC	89.17 (0.090)	6.57 (0.015)	6.72 (0.003)	0.08
		RF	89.38 (0.082)	6.55 (0.015)	6.72 (0.003)	0.08

Table 9. Comparison of relative frequencies in the estimation of $\hat{p}$ by BIC and AICc. The values correspond to a Monte Carlo simulation with 10,000 replications. Two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 100$. Nominal coverage 95%.

Factor	$\hat{p}=1$	$\hat{p}=2$	$\hat{p}=3$	$\hat{p}=4$	$\hat{p}=5$	$\hat{p}=6$
BIC						
F1	0.23	82.47	10.15	3.94	1.85	1.36
F2	1.84	78.47	12.32	4.01	1.95	1.41
AICc						
F1	0.02	36.66	16.17	13.50	14.13	19.52
F2	0.22	33.94	17.52	14.28	13.96	20.08

5. Empirical Example

As an application, we employ data of industrial production (486 seasonally adjusted monthly observations of the Industrial Production Index, IPI, from January, 1975, to June, 2015) in 13 European countries. These include Austria, Denmark, Finland, France, Germany, Italy, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, and the U.K. Other European countries with available data have been excluded for having small cross correlations with the former. See Figure 1 for a graph of the series included in the analysis.

In order to compare the results of the corrections we start with a rolling window of length $T = 50$, and forecast from $h = 1$ to $h = 12$ steps ahead. This means that, for the vector of 13 countries, the first window starts from the first observation in the data-set (January, 1975), until $T = 50$ (February, 1979). We work with this window to extract common factors[8], specify an AR model for each factor, and generate forecasts for the next 12 observations (March, 1979, to February, 1980). Repeating this process to the last window (from May, 2010, to June, 2014), we obtain 424 one- to twelve-step-ahead forecasts. The AR model for each factor is selected in each window, employing BIC. The prediction intervals will have 95% nominal coverage rates. We also performed these estimations employing longer windows of time $T = 100$ and $T = 200$, particularly to show how coverage rates C_m are linked to T in this data-set.

Some additional features outside the scope of the simulations of the previous sections help improve forecasts (equally for *none*, *BC*, and *RF*) in this application. Outliers are intervened beforehand using the statistical software TRAMO, through its Matlab interface. Figure 2 shows the series after intervening outliers. Furthermore, we obtained an important improvement of using three rather than two common factor to reduce the dimension of this dataset. Last, while in the simulations we know that the specific factors are white noise, in this practical application these are modeled as AR when necessary.

The analysis is performed for the logarithm of the series, but this transformation does not affect the conclusions.

To compare the results of using *none*, *BC*, and *RF* as small sample bias correction methods in the AR models for the common factors, we present actual coverage rates C_m, mean interval lengths L_m, and Mean Absolute Errors MAE. The MAE is a forecasting accuracy metric that helps evaluate which alternative forecast renders results closer to the observed values. It can be calculated in the following way,

$$MAE^j = \frac{1}{W} \sum_{w=1}^{W} \left(\frac{1}{13} \sum_{i=1}^{13} |(y_{i,z} - \hat{y}^j_{i,z})| \right), \tag{12}$$

8 For an assessment of the factors' loads see Figure 3a in Sub-Section 5.1.

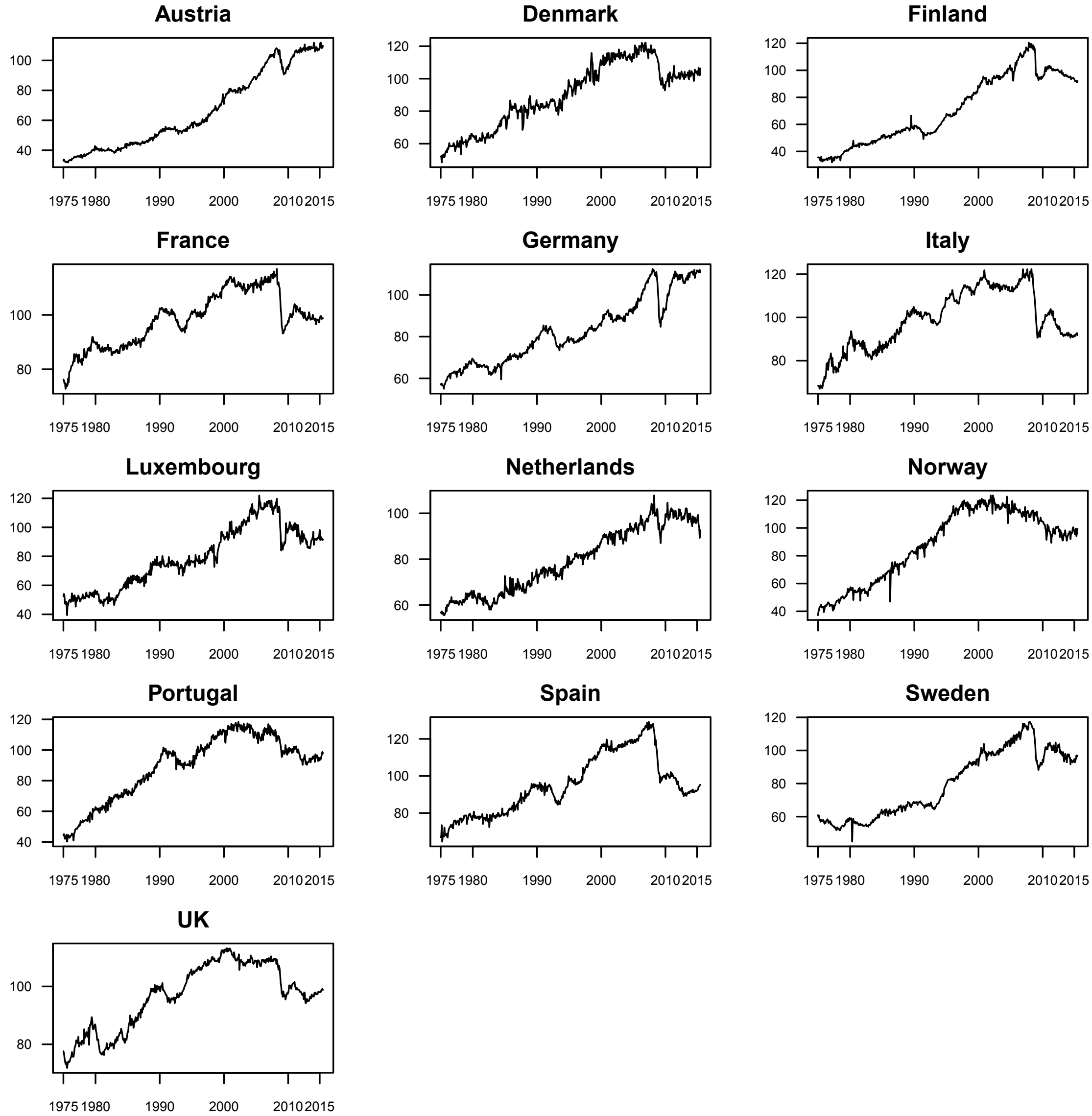


Figure 1. Industrial Production Index. Source: OECD.Stat. January, 1975 to June, 2015.

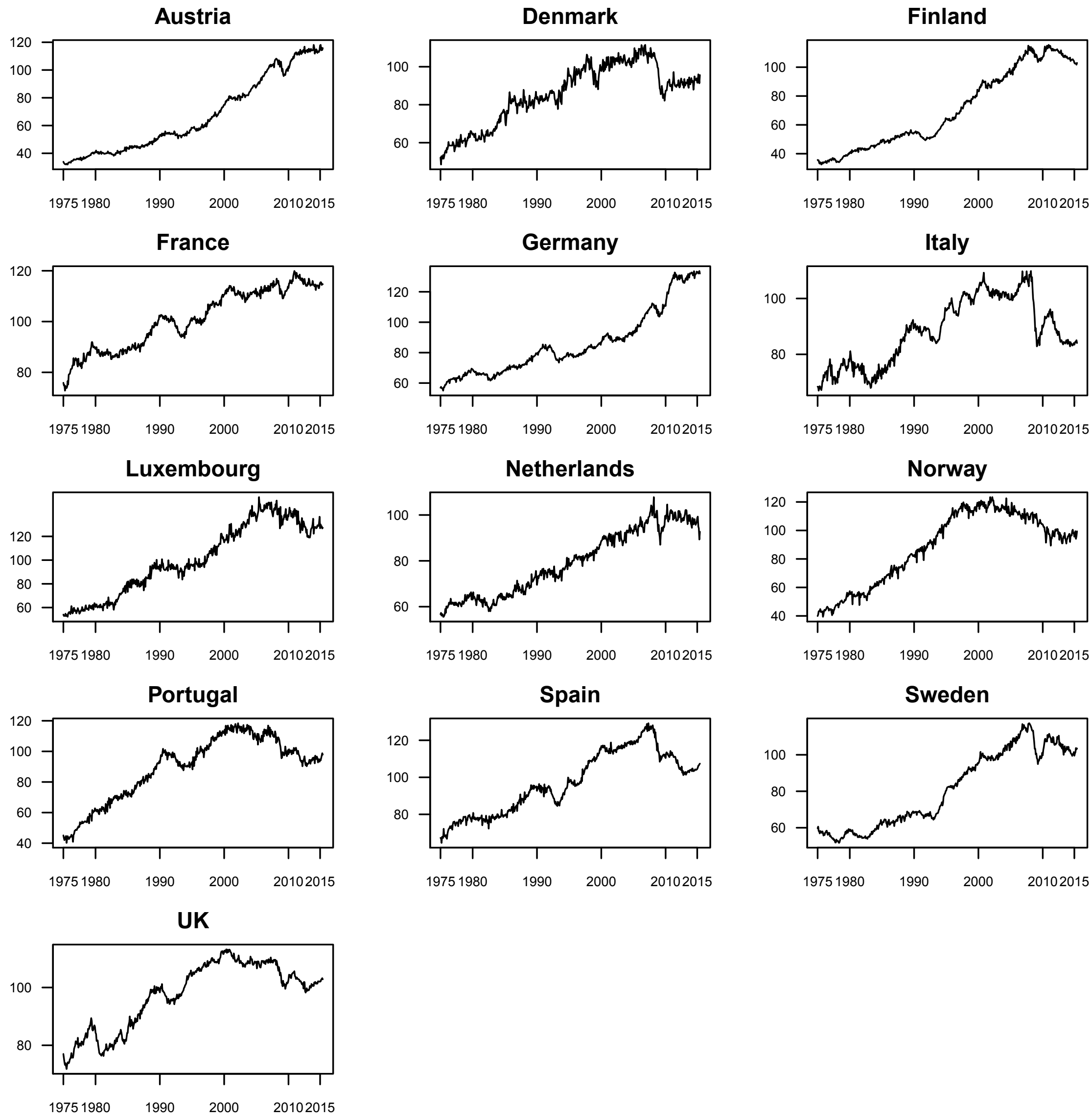


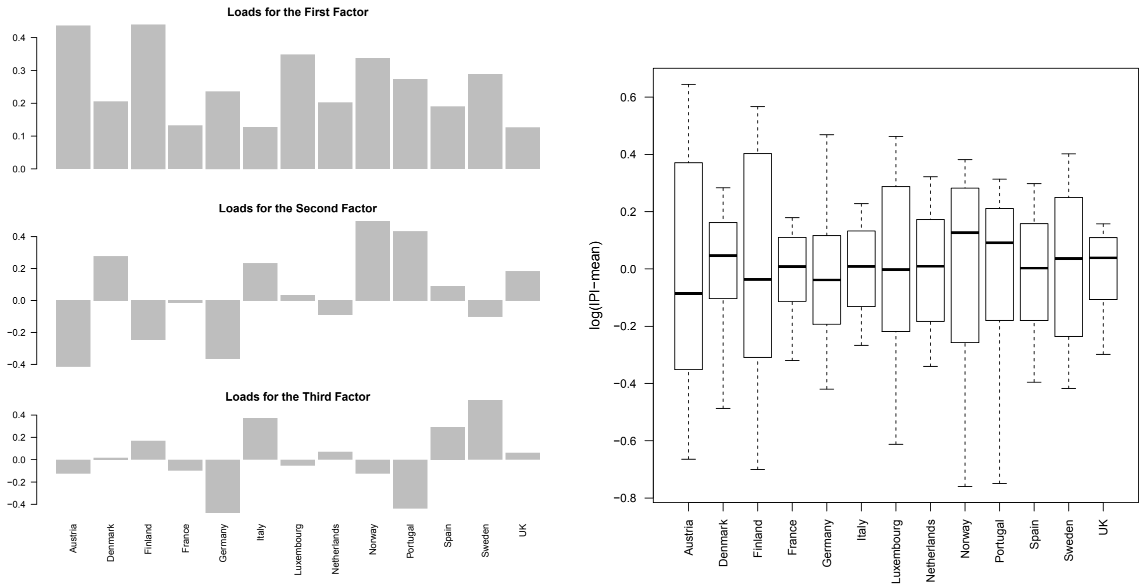
Figure 2. Industrial Production Index. Source: OECD.Stat. January, 1975 to June, 2015. Outliers intervention performed using TRAMO.

where W is the number of months in the out of sample period (the total number of rolling windows), $i = 1, ..., n$ the series included (in this case we have $n = 13$), and $j = \{none, BC, RF\}$. It is calculated for each forecasting horizon h.

5.1. Description of Loads

A feature of interest in the empirical estimation are the factors' loads. Because we are working with rolling windows (of diverse length T), loads are estimated together with the unobserved common factors in each window, and may change from on window to the next. For this reason, in Figure 3a we present the loads we would obtain for the whole dataset instead of any particular window of time; we do this to get an approximate representation of the matrix of weights. We also include box plots of the logarithm of centred IPI for the countries in this study, to identify similarities and differences in the distributions by country.

Oftentimes it is possible to visually find associations between loads and patterns or groupings in the data. The estimated weights for the first factor are highly associated to the variance of IPI in each country (see Table 10). The weights for the second factor distinguish two groups of countries: Denmark, Italy, Norway, Portugal, and the UK on the one hand, and Austria, Germany and, to a lesser degree, Finland, on the other hand. Last, the weights for the third factor (which, of course, contributes less to the total variability of the data than the other two) separate Germany and Portugal from Italy, Spain and Sweden.



(a) Loads corresponding to three unobserved common factors.

(b) Boxplots of the logarithm of the centred IPI, by country.

Figure 3. IPI complete dataset January, 1975, to June, 2015 ($T = 485$). Outliers intervened in the original series using TRAMO.

5.2. Results

In Tables 11 to 13 we present the average of the results for the 13 series, and in Tables 14 to 16 we present detailed results for four countries selected to represent diversity in coverage levels.

There are several findings. Firstly, for $T = 50$, the performance of the prediction intervals is short of the 95% nominal coverage. In this regard, Clements and Kim (2007) explain that a small-sample deterioration of the results of high-order models (they employ an AR(6) for US industrial production

Table 10. IPI complete dataset January, 1975, to June, 2015 ($T = 485$). Loads of the first common factor and variances for 13 European countries.

	Loads first factor	Variances IPI
Austria	0.44	0.16
Denmark	0.20	0.04
Finland	0.44	0.16
France	0.13	0.01
Germany	0.24	0.05
Italy	0.13	0.02
Luxembourg	0.35	0.10
Netherlands	0.20	0.03
Norway	0.34	0.10
Portugal	0.27	0.07
Spain	0.19	0.03
Sweden	0.29	0.07
UK	0.13	0.01

data) in comparison to low-order models (like those employed in simulations) is to be expected. The coverage, C_m, for this empirical example, is highly responsive to the size of the historical data (T) considered in the rolling windows: while for $T = 50$ we obtain C_m deteriorates to $C_m = 56.50\%$ ($h = 12$, *none*), Table 13 shows that, for $T = 200$, C_m is closer to the 95% nominal coverage (the worst coverage is $C_m = 78.50\%$, for $h = 12$ in *none*). For $h = 1$, comparing Tables 11, 12, and 13 we can see that the mean coverage, C_m, increases from 89.22 (averaging *none*, *BC*, and *RF* for $h = 1$) for $T = 50$, to 92.31 for $T = 100$, and almost reaches the nominal value for $T = 200$ (94.74 on average for $h = 1$). In other words, large sample sizes, though not always available in practice, contribute to more accurate forecasting intervals.

Secondly, in line with the results obtained in simulations, for one-step-ahead forecasts the improvements of *BC* and *RF* appear small; however, as the forecasting horizon increases, C_m and MAE reveal an evident advantage of employing the corrections, especially *RF*. For *RF*, the advantage in comparison to *none* reaches up to 10.38 percentage points (see Table 11, $h = 12$). Last, interval lengths L_m tend to be greater for *BC* and *RF* than for *none*.

Table 11. IPI forecasting results for 13 European countries. The average of the series is obtained for Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. $T = 50$.

Horizon	Correction	C_m (se)	MAE (se)	L_m (se)
h=1	none	88.82 (1.53)	1.47 (0.02)	6.03 (0.06)
	BC	89.56 (1.48)	1.46 (0.02)	6.07 (0.06)
	RF	89.29 (1.50)	1.46 (0.02)	6.06 (0.06)
h=6	none	71.74 (2.17)	2.78 (0.07)	7.67 (0.11)
	BC	76.50 (2.05)	2.65 (0.06)	8.17 (0.13)
	RF	77.60 (2.02)	2.58 (0.06)	8.21 (0.13)
h=12	none	56.50 (2.37)	4.29 (0.14)	8.46 (0.17)
	BC	64.90 (2.28)	4.12 (0.13)	9.68 (0.21)
	RF	66.52 (2.26)	3.85 (0.10)	9.74 (0.20)

Contrary to C_m, for this application the MAE does not seem to respond as much to sample size. This may seem odd, but it must be considered that the number of windows included in the estimation is smaller for Tables 12 and 13 because they have longer historical data-sets, than in Table 11.

In Tables 14 to 16 we present the results for Denmark, Finland, Luxembourg and Spain. Finland is the country with the highest coverage C_m in the short term while Luxembourg has the lowest

Table 12. IPI forecasting results for 13 European countries. The average of the series is obtained for Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. Rolling windows of size $T = 100$.

Horizon	Correction	C_m (se)	MAE (se)	L_m (se)
h=1	none	92.20 (1.37)	1.47 (0.02)	6.59 (0.05)
	BC	92.41 (1.34)	1.46 (0.02)	6.60 (0.05)
	RF	92.31 (1.36)	1.46 (0.02)	6.61 (0.05)
h=6	none	80.97 (2.01)	2.74 (0.06)	9.19 (0.11)
	BC	83.36 (1.91)	2.67 (0.06)	9.55 (0.12)
	RF	84.04 (1.88)	2.62 (0.06)	9.59 (0.12)
h=12	none	69.91 (2.34)	4.12 (0.11)	10.75 (0.17)
	BC	74.61 (2.22)	3.99 (0.11)	11.84 (0.20)
	RF	76.55 (2.17)	3.82 (0.09)	11.90 (0.19)

Table 13. IPI forecasting results for 13 European countries. The average of the series is obtained for Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. Rolling windows of size $T = 200$.

Horizon	Correction	C_m (se)	MAE (se)	L_m (se)
h=1	none	94.73 (1.31)	1.46 (0.03)	7.33 (0.07)
	BC	94.82 (1.30)	1.46 (0.03)	7.36 (0.07)
	RF	94.67 (1.32)	1.46 (0.03)	7.34 (0.06)
h=6	none	87.29 (1.98)	2.84 (0.08)	10.78 (0.11)
	BC	88.45 (1.88)	2.81 (0.07)	10.95 (0.11)
	RF	88.48 (1.88)	2.78 (0.07)	10.96 (0.11)
h=12	none	78.50 (2.46)	4.33 (0.13)	13.11 (0.16)
	BC	80.50 (2.38)	4.22 (0.12)	13.69 (0.17)
	RF	81.43 (2.32)	4.14 (0.12)	13.71 (0.17)

coverage for the first forecasting horizons. The results of Denmark and Spain are closer to the average results.

There may be some concerns regarding the way to best model this data that must be taken into account when interpreting the results. For instance, the models may be incorrectly specified (perhaps more sophisticated approaches should be used to model the factors), or there may be structural breaks in the data (in particular, this could be true for the windows containing the 2008 stock market crash) that are entirely ignored. However, these effects are out of the scope of this work and the empirical illustration still serves the purpose of demonstrating how bias-correcting the models for the common factors improves forecasting outcomes.

6. Concluding Remarks

Following Clements and Kim (2007) we have studied the behavior, in small samples, of three estimators for the AR parameters in a context of highly persistent models. Taking the applications of the methodology one step further, we have employed it in AR models for common factors, when we believe there are underlying unobserved factors driving the behavior of several time series. In all the cases we use the same bootstrap procedure to obtain prediction intervals (Alonso et al., 2008), so the only divergence originates in the estimation of the aforementioned AR parameters.

To evaluate this methodology, we carried out several Monte Carlo simulations, with alternative settings. These consisted of alternative sample sizes (in the time dimension) $T = 50, 100, 200$, different models for the behavior of the common factors (AR(1) and AR(2) factors), and various assumptions in regard to the information and tools available to the researcher, such as previous knowledge (or not) of the number of common factors to obtain and their AR order, and employing AICc or BIC criteria to select p, as well as the possibility of having non-Gaussian residuals.

Table 14. IPI results for four of the countries. Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. $T = 50$.

Horizon	Correction	C_m (se)				MAE (se)				L_m (se)			
		Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain
h=1	none	87.47 (1.61)	90.54 (1.42)	85.58 (1.71)	86.76 (1.65)	2.00 (0.08)	1.04 (0.04)	2.72 (0.11)	1.37 (0.05)	8.24 (0.08)	4.76 (0.06)	10.26 (0.11)	5.67 (0.04)
	BC	88.89 (1.53)	91.49 (1.36)	85.34 (1.72)	87.23 (1.62)	1.99 (0.08)	1.02 (0.04)	2.73 (0.11)	1.36 (0.05)	8.25 (0.08)	4.79 (0.06)	10.36 (0.11)	5.69 (0.04)
	RF	88.18 (1.57)	92.43 (1.29)	84.87 (1.74)	88.18 (1.57)	2.00 (0.08)	1.01 (0.04)	2.70 (0.11)	1.35 (0.05)	8.27 (0.08)	4.81 (0.06)	10.39 (0.11)	5.71 (0.04)
h=6	none	75.65 (2.09)	75.89 (2.08)	74.23 (2.13)	73.29 (2.15)	3.60 (0.14)	2.15 (0.09)	4.24 (0.18)	2.70 (0.13)	10.37 (0.14)	6.31 (0.10)	12.32 (0.16)	7.18 (0.10)
	BC	78.25 (2.01)	79.91 (1.95)	78.72 (1.99)	76.36 (2.07)	3.43 (0.13)	2.02 (0.08)	4.08 (0.18)	2.63 (0.12)	11.13 (0.17)	6.79 (0.11)	13.10 (0.18)	7.63 (0.12)
	RF	77.78 (2.02)	82.27 (1.86)	81.09 (1.91)	76.60 (2.06)	3.44 (0.14)	1.95 (0.08)	3.89 (0.15)	2.53 (0.11)	11.20 (0.17)	6.82 (0.11)	13.31 (0.18)	7.61 (0.11)
h=12	none	60.99 (2.37)	62.88 (2.35)	60.05 (2.38)	60.05 (2.38)	5.16 (0.24)	3.44 (0.18)	6.12 (0.27)	4.36 (0.23)	11.07 (0.17)	7.10 (0.17)	13.17 (0.18)	7.99 (0.16)
	BC	68.32 (2.26)	69.27 (2.25)	70.21 (2.23)	65.48 (2.31)	5.03 (0.23)	3.25 (0.16)	5.91 (0.27)	4.27 (0.23)	12.72 (0.25)	8.22 (0.21)	14.80 (0.23)	9.14 (0.22)
	RF	69.98 (2.23)	71.87 (2.19)	73.29 (2.15)	67.38 (2.28)	4.84 (0.21)	3.03 (0.13)	5.28 (0.19)	3.84 (0.18)	12.88 (0.25)	8.33 (0.18)	15.18 (0.24)	9.08 (0.19)

Table 15. IPI forecasting results for four of the countries. Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. Rolling windows of size $T = 100$.

Horizon	Correction	C_m (se)				MAE (se)				L_m (se)			
		Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain
h=1	none	90.35 (1.53)	95.17 (1.11)	86.06 (1.80)	89.54 (1.59)	2.07 (0.09)	1.03 (0.04)	3.00 (0.12)	1.33 (0.06)	8.91 (0.06)	5.32 (0.07)	11.21 (0.08)	6.12 (0.04)
	BC	90.08 (1.55)	96.51 (0.95)	84.45 (1.88)	90.62 (1.51)	2.07 (0.09)	1.00 (0.04)	3.00 (0.12)	1.32 (0.06)	8.98 (0.06)	5.40 (0.07)	11.21 (0.08)	6.15 (0.04)
	RF	88.74 (1.64)	95.98 (1.02)	86.86 (1.75)	90.35 (1.53)	2.06 (0.09)	1.01 (0.04)	3.00 (0.12)	1.31 (0.06)	8.92 (0.06)	5.37 (0.07)	11.27 (0.08)	6.14 (0.04)
h=6	none	80.43 (2.06)	83.65 (1.92)	77.75 (2.16)	76.68 (2.19)	3.64 (0.15)	2.33 (0.10)	4.59 (0.18)	3.02 (0.13)	12.16 (0.10)	8.18 (0.13)	14.48 (0.15)	8.56 (0.09)
	BC	83.38 (1.93)	86.33 (1.78)	81.77 (2.00)	78.82 (2.12)	3.54 (0.15)	2.20 (0.09)	4.56 (0.18)	2.89 (0.13)	12.68 (0.11)	8.57 (0.13)	15.05 (0.16)	8.85 (0.10)
	RF	82.84 (1.95)	87.94 (1.69)	81.23 (2.02)	79.62 (2.09)	3.53 (0.15)	2.12 (0.08)	4.47 (0.18)	2.79 (0.12)	12.72 (0.11)	8.59 (0.13)	15.27 (0.17)	8.87 (0.09)
h=12	none	75.60 (2.23)	67.02 (2.44)	67.29 (2.43)	56.03 (2.57)	4.99 (0.22)	4.01 (0.16)	6.47 (0.24)	5.05 (0.20)	13.79 (0.16)	9.94 (0.19)	16.36 (0.19)	10.18 (0.16)
	BC	79.89 (2.08)	73.99 (2.27)	73.19 (2.30)	64.34 (2.48)	4.76 (0.21)	3.77 (0.15)	6.40 (0.23)	4.81 (0.20)	15.08 (0.19)	11.10 (0.21)	17.92 (0.22)	11.12 (0.18)
	RF	80.70 (2.05)	77.48 (2.17)	78.28 (2.14)	67.02 (2.44)	4.57 (0.21)	3.56 (0.14)	6.12 (0.21)	4.53 (0.18)	15.17 (0.19)	11.11 (0.20)	18.33 (0.24)	11.16 (0.17)

Table 16. IPI forecasting results for four of the countries. Interval Coverage C_m (in %), Mean Absolute Error MAE, and Interval's Length L_m. Standard Errors are provided between parenthesis. Rolling windows of size $T = 200$.

Horizon	Correction	C_m (se)				MAE (se)				L_m (se)			
		Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain	Denmark	Finland	Luxembourg	Spain
h=1	none	91.58 (1.68)	97.44 (0.96)	90.11 (1.81)	94.14 (1.42)	2.08 (0.10)	1.11 (0.06)	3.00 (0.16)	1.25 (0.06)	9.83 (0.09)	6.57 (0.08)	12.40 (0.09)	6.96 (0.04)
	BC	91.94 (1.65)	97.44 (0.96)	89.01 (1.90)	95.24 (1.29)	2.08 (0.10)	1.10 (0.05)	2.97 (0.16)	1.23 (0.06)	9.85 (0.08)	6.59 (0.08)	12.51 (0.09)	6.96 (0.05)
	RF	91.58 (1.68)	97.44 (0.96)	89.74 (1.84)	95.24 (1.29)	2.08 (0.10)	1.10 (0.05)	2.99 (0.16)	1.23 (0.06)	9.83 (0.09)	6.57 (0.08)	12.44 (0.09)	7.01 (0.05)
h=6	none	88.64 (1.92)	89.01 (1.90)	79.85 (2.43)	79.12 (2.46)	3.38 (0.20)	2.82 (0.13)	5.12 (0.26)	3.02 (0.16)	13.61 (0.08)	10.86 (0.17)	17.29 (0.16)	9.99 (0.06)
	BC	86.81 (2.05)	92.67 (1.58)	83.15 (2.27)	79.49 (2.45)	3.40 (0.20)	2.66 (0.12)	4.92 (0.25)	2.95 (0.15)	13.71 (0.07)	11.14 (0.16)	17.51 (0.16)	10.14 (0.06)
	RF	87.18 (2.03)	93.77 (1.47)	83.15 (2.27)	82.42 (2.31)	3.39 (0.20)	2.57 (0.11)	4.89 (0.25)	2.85 (0.15)	13.74 (0.08)	11.19 (0.17)	17.47 (0.16)	10.10 (0.06)
h=12	none	85.71 (2.12)	71.06 (2.75)	73.26 (2.68)	68.86 (2.81)	4.37 (0.30)	4.83 (0.25)	7.41 (0.38)	5.04 (0.25)	15.09 (0.07)	14.05 (0.24)	20.83 (0.23)	12.13 (0.10)
	BC	86.08 (2.10)	77.66 (2.53)	79.49 (2.45)	71.06 (2.75)	4.41 (0.30)	4.55 (0.22)	6.86 (0.36)	4.84 (0.24)	15.53 (0.08)	14.97 (0.27)	21.65 (0.24)	12.65 (0.10)
	RF	85.71 (2.12)	77.29 (2.54)	80.22 (2.42)	71.79 (2.73)	4.36 (0.30)	4.42 (0.20)	6.63 (0.36)	4.69 (0.24)	15.62 (0.08)	14.98 (0.26)	21.68 (0.23)	12.65 (0.10)

Our most important finding is that in all the settings considered, the two techniques *BC* and *RF* succeed at obtaining improved coverage rates in comparison with the situation when no correction is performed. Furthermore, *RF* tends to be the most advantageous.

Another outcome of the simulations is that, as expected, the smaller the sample (T), the greater the improvement due to bias correction. Therefore, it is more effective to use correction techniques when the sample size is small (which we have represented with $T = 50$) than for larger samples (in particular, we have worked with $T = 200$).

Additionally, the edge of the techniques employed over *none* augments for longer forecasting horizons, another result in line with Clements and Kim (2007).

Lastly, though the empirical results turn out to be rather modest measured by coverage rates, still reveal large differences in performance of the corrected methods vs. *none*. In agreement with the simulations outcomes, the improvements are more noticeable as the forecasting horizon increases.

Possible extensions include exploring the bias when the common factors follow alternative specifications. For instance, MA terms could be included to the AR models hereby studied. Also, seasonality may be modelled if needed. Another option would be to include VAR specifications for the common factors instead of AR.

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Appendix A. Details about the results for the common factors

Because we work with simulated data, we generate and know the values of the underlying factors, contrary to the situation of working with empirical data. In this section we take advantage of this setting to understand better the circumstances surrounding the estimation of the models for the common factors F_1 and F_2. We provide detail for the case of two factors that follow AR(1) processes, both r and p are assumed to be known.

In our simulation, it is straight forward to check the bias for factors that are AR(1). This is done in Table A1. The bias of *BC* and *RF* is much smaller than the one for *none*. Notice however that the estimation rendered by *none* gets closer to the true value of the coefficients ϕ_{F1}, ϕ_{F2} as the sample gets larger. Thus, the emphasis in that the correction techniques employed in this work are particularly beneficial for small samples.

Table A1. Bias for common factors following AR(1) processes with normal errors. Between parenthesis, the variance of the AR estimated coefficients. Monte Carlo simulations of model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. 10,000 MC replications.

Factor	Correction	T=50	T=100	T=200
F1	none	0.086 (0.006)	0.045 (0.002)	0.023 (0.001)
	BC	0.028 (0.005)	0.011 (0.002)	0.003 (0.001)
	RF	0.018 (0.005)	0.005 (0.002)	-0.001 (0.001)
F2	none	0.157 (0.013)	0.078 (0.005)	0.040 (0.002)
	BC	0.092 (0.015)	0.042 (0.005)	0.021 (0.002)
	RF	0.078 (0.017)	0.037 (0.005)	0.021 (0.002)

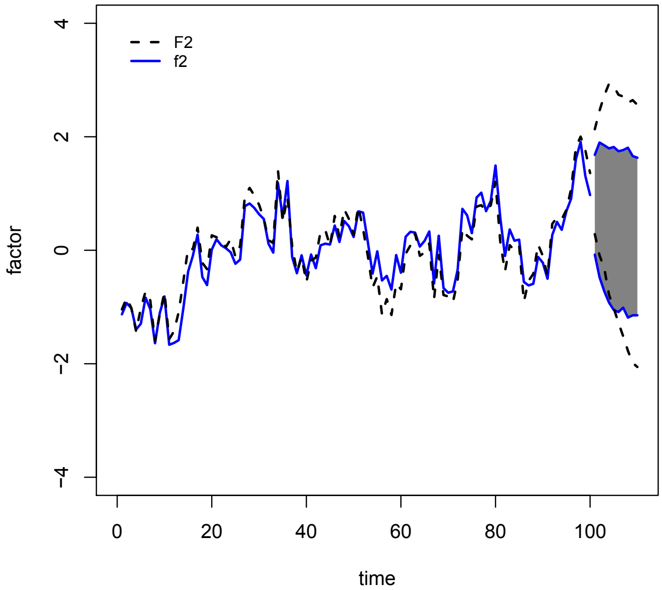

Figure A1. Example of the estimation of F_2 for a window with $T = 100$. *Simulation of model with coefficients* $\phi_{F1} = 0.975, \phi_{F2} = 0.90$.

Tables A2, A3, and A4 present the results of the same simulation, this time for the factors (instead of the selected series) for each indicated sample size. We can see that the interval coverage C_m is oftentimes far from the theoretical 95%, especially for F_2. Notwithstanding, the performance in terms of coverage of the series (studied in Section 4.1) is much better. Take for instance the estimation of F_2 of a sample of size $T = 100$. In Figure A1 we present F_2 and its estimate f_2 for a random sample, of the model in which factors follow AR(1) processes. Notice that for the last "observed" value (for $time = 100$), F_2 and the factor estimation f_2 are slightly different; in this case f_2 is smaller than the actual value for F_2. As a consequence, the forecasting interval for f_2 does not match exactly what would be the interval for the actual values of F_2.[9] In this case the forecasting interval of f_2 (blue solid lines) is smaller than the equivalent interval for the continuations (black dotted lines). This breach has a negative effect in C_m for the factor, as we can see in Table A3, with coverage values around 86.50% for $h = 1$, and even more when the samples are smaller like in Table A2.

Besides this point, we observe the same patterns for the factors than for the series. This is to be expected given that, as previously explained, the time series are mainly the product of the common factors times some weights.

[9] The interval for the actual values of F_2 is obtained using continuations given the known value of $F_2(T = 100)$ and the known value ϕ_{F2}.

Table A2. Results of Monte Carlo simulation, 10,000 replications. Two common factors following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 50$. Nominal coverage 95%.

Factor	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
F1	h=1	none	90.38 (0.079)	3.73 (0.006)	3.88 (0.002)	0.09
		BC	90.68 (0.081)	3.78 (0.006)	3.88 (0.002)	0.07
		RF	90.94 (0.079)	3.78 (0.006)	3.88 (0.002)	0.07
	h=5	none	85.33 (0.087)	6.87 (0.012)	8.27 (0.003)	0.27
		BC	89.02 (0.078)	7.69 (0.013)	8.27 (0.003)	0.13
		RF	90.76 (0.071)	7.87 (0.014)	8.27 (0.003)	0.09
	h=10	none	79.38 (0.116)	8.16 (0.020)	11.01 (0.005)	0.42
		BC	85.93 (0.111)	10.05 (0.025)	11.01 (0.005)	0.18
		RF	89.21 (0.100)	10.51 (0.025)	11.01 (0.005)	0.11
F2	h=1	none	82.34 (0.198)	2.02 (0.003)	1.94 (0.001)	0.17
		BC	82.42 (0.200)	2.03 (0.003)	1.94 (0.001)	0.18
		RF	82.40 (0.201)	2.03 (0.004)	1.94 (0.001)	0.18
	h=5	none	84.28 (0.097)	3.02 (0.006)	3.60 (0.002)	0.27
		BC	87.50 (0.093)	3.35 (0.008)	3.60 (0.002)	0.15
		RF	87.99 (0.094)	3.46 (0.008)	3.60 (0.002)	0.11
	h=10	none	82.28 (0.096)	3.18 (0.008)	4.17 (0.002)	0.37
		BC	86.58 (0.099)	3.77 (0.012)	4.17 (0.002)	0.18
		RF	87.44 (0.101)	4.01 (0.014)	4.17 (0.002)	0.12

Table A3. Results of Monte Carlo simulation, 10,000 replications. Two common factors following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 100$. Nominal coverage 95%.

Factor	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
F1	h=1	none	92.85 (0.040)	3.83 (0.004)	3.88 (0.002)	0.04
		BC	92.92 (0.040)	3.85 (0.004)	3.88 (0.002)	0.03
		RF	93.08 (0.040)	3.86 (0.004)	3.88 (0.002)	0.03
	h=5	none	90.23 (0.054)	7.52 (0.009)	8.27 (0.003)	0.14
		BC	92.18 (0.046)	8.06 (0.010)	8.27 (0.003)	0.06
		RF	92.85 (0.043)	8.18 (0.010)	8.27 (0.003)	0.03
	h=10	none	86.75 (0.080)	9.32 (0.016)	11.02 (0.005)	0.24
		BC	90.60 (0.070)	10.68 (0.019)	11.02 (0.005)	0.08
		RF	91.90 (0.064)	10.98 (0.019)	11.02 (0.005)	0.04
F2	h=1	none	86.50 (0.158)	2.02 (0.002)	1.94 (0.001)	0.13
		BC	86.49 (0.160)	2.02 (0.002)	1.94 (0.001)	0.13
		RF	86.38 (0.161)	2.03 (0.002)	1.94 (0.001)	0.13
	h=5	none	89.68 (0.062)	3.32 (0.005)	3.60 (0.001)	0.13
		BC	91.39 (0.058)	3.53 (0.005)	3.60 (0.001)	0.06
		RF	91.40 (0.059)	3.57 (0.006)	3.60 (0.001)	0.05
	h=10	none	88.38 (0.066)	3.61 (0.006)	4.17 (0.002)	0.20
		BC	90.96 (0.063)	4.01 (0.008)	4.17 (0.002)	0.08
		RF	91.11 (0.065)	4.10 (0.009)	4.17 (0.002)	0.06

Table A4. Results of Monte Carlo simulation, 10,000 replications. Two common factors following AR(1) models with normal errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 200$. Nominal coverage 95%.

Factor	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
F1	h=1	none	93.95 (0.024)	3.89 (0.003)	3.88 (0.002)	0.01
		BC	93.96 (0.024)	3.89 (0.003)	3.88 (0.002)	0.01
		RF	93.97 (0.025)	3.90 (0.003)	3.88 (0.002)	0.01
	h=5	none	92.76 (0.032)	7.92 (0.007)	8.27 (0.003)	0.07
		BC	93.73 (0.028)	8.25 (0.007)	8.27 (0.003)	0.02
		RF	93.88 (0.027)	8.32 (0.007)	8.27 (0.003)	0.02
	h=10	none	91.06 (0.048)	10.13 (0.012)	11.01 (0.005)	0.12
		BC	93.04 (0.041)	11.00 (0.013)	11.01 (0.005)	0.02
		RF	93.40 (0.040)	11.21 (0.014)	11.01 (0.005)	0.04
F2	h=1	none	89.96 (0.111)	2.01 (0.002)	1.94 (0.001)	0.09
		BC	89.94 (0.112)	2.02 (0.002)	1.94 (0.001)	0.09
		RF	89.97 (0.112)	2.02 (0.002)	1.94 (0.001)	0.09
	h=5	none	92.59 (0.038)	3.50 (0.003)	3.59 (0.002)	0.05
		BC	93.41 (0.035)	3.61 (0.004)	3.59 (0.002)	0.02
		RF	93.41 (0.036)	3.62 (0.004)	3.59 (0.002)	0.02
	h=10	none	91.81 (0.042)	3.91 (0.005)	4.17 (0.002)	0.10
		BC	93.19 (0.039)	4.13 (0.006)	4.17 (0.002)	0.03
		RF	93.18 (0.040)	4.14 (0.006)	4.17 (0.002)	0.03

Appendix B. Results for Non-Gaussian Errors

In this section we re-run the simulations for AR(1) common factors introducing non normal errors. In particular, the innovations $\eta_{i,t}$ (in (3)) follow a centred $\chi^2(5)$ distribution (like Clements and Kim, 2007). The main difference of this distribution with the normal is that it is not symmetrical. Table B1 is the analogous to Table A1. Notice that the bias does not seem to worsen with the new distribution of $\eta_{i,t}$. *BC* and *RF* continue to improve upon *none*, in a similar measure to the case of normally distributed errors. Still, as expected, the estimation without any bias correction gets closer to the true AR coefficients (ϕ_{F1}, ϕ_{F2}) as the sample (in the time dimension, T) gets larger.

Table B1. Bias for common factors following AR(1) processes with centred $\chi^2(5)$ errors. Between parenthesis, the variance of the AR estimated coefficients. Monte Carlo simulations of model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. 10,000 MC replications.

factor	method	T=50	T=100	T=200
F1	none	0.083 (0.004)	0.042 (0.001)	0.022 (0.001)
	BC	0.015 (0.003)	0.004 (0.001)	0.001 (0.001)
	RF	0.002 (0.003)	-0.003 (0.001)	-0.004 (0.001)
F2	none	0.177 (0.013)	0.078 (0.004)	0.035 (0.001)
	BC	0.111 (0.015)	0.042 (0.004)	0.016 (0.002)
	RF	0.096 (0.017)	0.036 (0.005)	0.015 (0.002)

The results in terms of coverage, interval length, and CQ_m, are similar to those for the process with normal errors, revealing that the bias-corrections for the underlying non observed factors improve forecasting results even when they are not normally distributed. For the five selected series, results are presented in Tables B2-B4.

Table B2. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with centred $\chi^2(5)$ errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 50$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	91.89 (0.044)	4.60 (0.008)	4.79 (0.003)	0.07
		BC	92.26 (0.042)	4.64 (0.008)	4.79 (0.003)	0.06
		RF	92.34 (0.041)	4.66 (0.008)	4.79 (0.003)	0.06
	h=5	none	85.02 (0.088)	7.36 (0.015)	9.05 (0.004)	0.29
		BC	88.64 (0.081)	8.31 (0.018)	9.05 (0.004)	0.15
		RF	89.44 (0.079)	8.58 (0.019)	9.05 (0.004)	0.11
	h=10	none	81.15 (0.106)	8.23 (0.021)	10.99 (0.005)	0.40
		BC	87.09 (0.103)	10.24 (0.030)	10.99 (0.005)	0.15
		RF	88.61 (0.100)	10.86 (0.032)	10.99 (0.005)	0.08
Y2	h=1	none	92.26 (0.056)	2.29 (0.004)	2.35 (0.001)	0.05
		BC	93.05 (0.048)	2.31 (0.004)	2.35 (0.001)	0.04
		RF	93.23 (0.047)	2.32 (0.004)	2.35 (0.001)	0.03
	h=5	none	82.44 (0.105)	3.93 (0.008)	5.02 (0.002)	0.35
		BC	87.93 (0.091)	4.45 (0.010)	5.02 (0.002)	0.19
		RF	89.08 (0.089)	4.57 (0.010)	5.02 (0.002)	0.15
	h=10	none	75.73 (0.129)	4.53 (0.012)	6.70 (0.003)	0.53
		BC	84.74 (0.122)	5.67 (0.016)	6.70 (0.003)	0.26
		RF	86.88 (0.119)	5.99 (0.017)	6.70 (0.003)	0.19
Y5	h=1	none	91.75 (0.042)	5.98 (0.010)	6.25 (0.003)	0.08
		BC	92.23 (0.039)	6.05 (0.010)	6.25 (0.003)	0.06
		RF	92.35 (0.038)	6.07 (0.010)	6.25 (0.003)	0.06
	h=5	none	83.94 (0.093)	9.77 (0.020)	12.28 (0.005)	0.32
		BC	88.36 (0.082)	11.04 (0.023)	12.28 (0.005)	0.17
		RF	89.25 (0.081)	11.37 (0.024)	12.28 (0.005)	0.13
	h=10	none	78.45 (0.116)	11.07 (0.029)	15.65 (0.007)	0.47
		BC	85.89 (0.112)	13.82 (0.039)	15.65 (0.007)	0.21
		RF	87.70 (0.109)	14.62 (0.042)	15.65 (0.007)	0.14
Y10	h=1	none	92.79 (0.051)	6.21 (0.011)	6.29 (0.004)	0.04
		BC	93.04 (0.049)	6.26 (0.011)	6.29 (0.004)	0.02
		RF	93.09 (0.048)	6.29 (0.011)	6.29 (0.004)	0.02
	h=5	none	85.78 (0.089)	9.92 (0.021)	11.92 (0.005)	0.26
		BC	88.89 (0.083)	11.20 (0.025)	11.92 (0.005)	0.12
		RF	89.62 (0.080)	11.57 (0.027)	11.92 (0.005)	0.09
	h=10	none	83.10 (0.102)	11.01 (0.028)	13.96 (0.006)	0.34
		BC	88.00 (0.101)	13.67 (0.041)	13.96 (0.006)	0.09
		RF	89.35 (0.095)	14.53 (0.045)	13.96 (0.006)	0.10
Y25	h=1	none	92.52 (0.053)	8.77 (0.015)	8.99 (0.005)	0.05
		BC	92.76 (0.051)	8.84 (0.015)	8.99 (0.005)	0.04
		RF	92.80 (0.050)	8.86 (0.015)	8.99 (0.005)	0.04
	h=5	none	85.56 (0.089)	14.22 (0.029)	17.13 (0.008)	0.27
		BC	88.77 (0.084)	16.06 (0.036)	17.13 (0.008)	0.13
		RF	89.52 (0.081)	16.58 (0.038)	17.13 (0.008)	0.09
	h=10	none	82.68 (0.105)	15.85 (0.040)	20.19 (0.009)	0.34
		BC	87.80 (0.103)	19.68 (0.058)	20.19 (0.009)	0.10
		RF	89.22 (0.097)	20.93 (0.064)	20.19 (0.009)	0.10

Table B3. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with centred $\chi^2(5)$ errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 100$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	93.27 (0.029)	4.68 (0.006)	4.79 (0.003)	0.04
		BC	93.36 (0.028)	4.70 (0.006)	4.79 (0.003)	0.04
		RF	93.41 (0.028)	4.71 (0.006)	4.79 (0.003)	0.03
	h=5	none	89.87 (0.055)	8.11 (0.011)	9.04 (0.004)	0.16
		BC	91.67 (0.050)	8.69 (0.013)	9.04 (0.004)	0.07
		RF	91.87 (0.051)	8.83 (0.013)	9.04 (0.004)	0.06
	h=10	none	87.47 (0.073)	9.36 (0.017)	10.98 (0.005)	0.23
		BC	90.73 (0.067)	10.65 (0.021)	10.98 (0.005)	0.07
		RF	91.22 (0.067)	10.99 (0.023)	10.98 (0.005)	0.04
Y2	h=1	none	93.88 (0.036)	2.33 (0.003)	2.34 (0.001)	0.02
		BC	94.17 (0.034)	2.34 (0.003)	2.34 (0.001)	0.01
		RF	94.29 (0.033)	2.35 (0.003)	2.34 (0.001)	0.01
	h=5	none	89.13 (0.065)	4.46 (0.006)	5.01 (0.002)	0.17
		BC	91.80 (0.054)	4.79 (0.007)	5.01 (0.002)	0.08
		RF	92.37 (0.052)	4.88 (0.007)	5.01 (0.002)	0.05
	h=10	none	85.04 (0.089)	5.44 (0.010)	6.69 (0.003)	0.29
		BC	90.05 (0.077)	6.28 (0.012)	6.69 (0.003)	0.11
		RF	91.10 (0.073)	6.49 (0.013)	6.69 (0.003)	0.07
Y5	h=1	none	93.24 (0.027)	6.11 (0.007)	6.25 (0.003)	0.04
		BC	93.38 (0.026)	6.14 (0.007)	6.25 (0.003)	0.03
		RF	93.43 (0.026)	6.15 (0.007)	6.25 (0.003)	0.03
	h=5	none	89.56 (0.057)	10.93 (0.015)	12.27 (0.006)	0.17
		BC	91.76 (0.049)	11.74 (0.017)	12.27 (0.006)	0.08
		RF	92.12 (0.048)	11.94 (0.017)	12.27 (0.006)	0.06
	h=10	none	86.32 (0.080)	12.99 (0.024)	15.63 (0.007)	0.26
		BC	90.43 (0.070)	14.88 (0.029)	15.63 (0.007)	0.10
		RF	91.19 (0.068)	15.37 (0.031)	15.63 (0.007)	0.06
Y10	h=1	none	94.15 (0.033)	6.30 (0.008)	6.29 (0.004)	0.01
		BC	94.18 (0.033)	6.32 (0.008)	6.29 (0.004)	0.01
		RF	94.15 (0.034)	6.33 (0.008)	6.29 (0.004)	0.02
	h=5	none	90.17 (0.057)	10.81 (0.016)	11.91 (0.005)	0.14
		BC	91.74 (0.054)	11.57 (0.017)	11.91 (0.005)	0.06
		RF	91.81 (0.055)	11.75 (0.018)	11.91 (0.005)	0.05
	h=10	none	88.42 (0.070)	12.24 (0.022)	13.95 (0.006)	0.19
		BC	91.01 (0.068)	13.85 (0.028)	13.95 (0.006)	0.05
		RF	91.29 (0.068)	14.26 (0.031)	13.95 (0.006)	0.06
Y25	h=1	none	93.90 (0.036)	8.89 (0.010)	8.99 (0.005)	0.02
		BC	93.92 (0.036)	8.92 (0.011)	8.99 (0.005)	0.02
		RF	93.87 (0.037)	8.93 (0.010)	8.99 (0.005)	0.02
	h=5	none	89.97 (0.058)	15.48 (0.022)	17.11 (0.008)	0.15
		BC	91.67 (0.054)	16.60 (0.025)	17.11 (0.008)	0.07
		RF	91.76 (0.054)	16.85 (0.026)	17.11 (0.008)	0.05
	h=10	none	88.19 (0.071)	17.62 (0.032)	20.18 (0.009)	0.20
		BC	90.97 (0.068)	19.98 (0.041)	20.18 (0.009)	0.05
		RF	91.28 (0.068)	20.59 (0.044)	20.18 (0.009)	0.06

Table B4. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created using two common factors, both following AR(1) models with centred $\chi^2(5)$ errors. Model with coefficients $\phi_{F1} = 0.975, \phi_{F2} = 0.90$. $T = 200$. Nominal coverage 95%.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	93.84 (0.022)	4.72 (0.004)	4.79 (0.003)	0.03
		BC	93.88 (0.021)	4.72 (0.004)	4.79 (0.003)	0.03
		RF	93.85 (0.022)	4.73 (0.004)	4.79 (0.003)	0.03
	h=5	none	92.35 (0.034)	8.55 (0.008)	9.05 (0.004)	0.08
		BC	93.21 (0.031)	8.88 (0.009)	9.05 (0.004)	0.04
		RF	93.22 (0.032)	8.92 (0.009)	9.05 (0.004)	0.03
	h=10	none	91.13 (0.046)	10.10 (0.013)	10.99 (0.005)	0.12
		BC	92.73 (0.042)	10.84 (0.014)	10.99 (0.005)	0.04
		RF	92.85 (0.042)	10.97 (0.015)	10.99 (0.005)	0.02
Y2	h=1	none	94.21 (0.029)	2.33 (0.002)	2.34 (0.001)	0.01
		BC	94.33 (0.029)	2.34 (0.002)	2.34 (0.001)	0.01
		RF	94.32 (0.029)	2.34 (0.002)	2.34 (0.001)	0.01
	h=5	none	92.25 (0.039)	4.75 (0.005)	5.01 (0.002)	0.08
		BC	93.45 (0.034)	4.96 (0.005)	5.01 (0.002)	0.03
		RF	93.58 (0.033)	5.00 (0.005)	5.01 (0.002)	0.02
	h=10	none	90.23 (0.056)	6.05 (0.008)	6.69 (0.003)	0.15
		BC	92.57 (0.048)	6.59 (0.009)	6.69 (0.003)	0.04
		RF	92.90 (0.046)	6.72 (0.009)	6.69 (0.003)	0.03
Y5	h=1	none	93.82 (0.020)	6.16 (0.005)	6.25 (0.003)	0.03
		BC	93.87 (0.020)	6.17 (0.005)	6.25 (0.003)	0.03
		RF	93.86 (0.020)	6.17 (0.005)	6.25 (0.003)	0.02
	h=5	none	92.38 (0.033)	11.61 (0.011)	12.27 (0.006)	0.08
		BC	93.37 (0.030)	12.09 (0.011)	12.27 (0.006)	0.03
		RF	93.44 (0.030)	12.17 (0.012)	12.27 (0.006)	0.02
	h=10	none	90.78 (0.048)	14.26 (0.018)	15.64 (0.007)	0.13
		BC	92.74 (0.043)	15.44 (0.020)	15.64 (0.007)	0.04
		RF	92.96 (0.042)	15.68 (0.021)	15.64 (0.007)	0.02
Y10	h=1	none	94.56 (0.026)	6.32 (0.006)	6.29 (0.004)	0.01
		BC	94.58 (0.026)	6.32 (0.006)	6.29 (0.004)	0.01
		RF	94.53 (0.027)	6.32 (0.006)	6.29 (0.004)	0.01
	h=5	none	92.41 (0.038)	11.31 (0.012)	11.92 (0.005)	0.08
		BC	93.19 (0.036)	11.72 (0.012)	11.92 (0.005)	0.04
		RF	93.15 (0.038)	11.76 (0.013)	11.92 (0.005)	0.03
	h=10	none	91.37 (0.047)	12.99 (0.017)	13.97 (0.006)	0.11
		BC	92.70 (0.045)	13.82 (0.019)	13.97 (0.006)	0.03
		RF	92.69 (0.047)	13.92 (0.020)	13.97 (0.006)	0.03
Y25	h=1	none	94.34 (0.028)	8.95 (0.008)	8.99 (0.005)	0.01
		BC	94.36 (0.028)	8.95 (0.008)	8.99 (0.005)	0.01
		RF	94.32 (0.029)	8.96 (0.008)	8.99 (0.005)	0.01
	h=5	none	92.27 (0.038)	16.22 (0.016)	17.12 (0.008)	0.08
		BC	93.09 (0.037)	16.79 (0.017)	17.12 (0.008)	0.04
		RF	93.07 (0.038)	16.86 (0.018)	17.12 (0.008)	0.04
	h=10	none	91.18 (0.048)	18.71 (0.024)	20.20 (0.009)	0.11
		BC	92.59 (0.046)	19.94 (0.028)	20.20 (0.009)	0.04
		RF	92.62 (0.048)	20.11 (0.028)	20.20 (0.009)	0.03

Appendix C. Number of Factors and AR Orders Unknown for T=50 and T=200

This section complements the results presented in Section 4.2, considering the alternative sample sizes $T = 50$ (Tables C1 and C2) and $T = 200$ (Tables C3 and C4). It continues to be the case that the indicators C_m, L_m, and CQ are equal or better for BC and RF than for *none*. Consistently with the results obtained in previous sections, the results are enhanced as the forecasting horizon h increases.

Additionally, for all the sample sizes (T), the results of employing BIC in the selection of p outperform those of employing AICc. The performance of these criteria is recorded in Tables C5 and C6.

Last, it must be acknowledged that there is a sharp deterioration of results when the time frame reduces to $T = 50$. In this regard, it should be reminded that the common factors are estimates themselves and their accuracy improves with the sample size.

Table C1. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created with both common factors following AR(2) processes with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 50$. Nominal coverage 95%. IC_3 used in the estimation of R, BIC used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	88.37 (0.071)	0.96 (0.001)	1.05 (0.000)	0.15
		BC	89.00 (0.067)	0.97 (0.002)	1.05 (0.000)	0.14
		RF	89.39 (0.065)	0.98 (0.002)	1.05 (0.000)	0.13
	h=5	none	80.58 (0.132)	2.67 (0.006)	3.29 (0.001)	0.34
		BC	84.91 (0.119)	2.97 (0.007)	3.29 (0.001)	0.20
		RF	86.27 (0.106)	2.95 (0.007)	3.29 (0.001)	0.19
	h=10	none	73.98 (0.158)	3.31 (0.010)	4.61 (0.002)	0.50
		BC	81.20 (0.155)	4.08 (0.013)	4.61 (0.002)	0.26
		RF	84.26 (0.128)	4.11 (0.013)	4.61 (0.002)	0.22
Y2	h=1	none	84.65 (0.078)	0.69 (0.001)	0.84 (0.000)	0.29
		BC	85.30 (0.075)	0.69 (0.001)	0.84 (0.000)	0.28
		RF	85.75 (0.071)	0.70 (0.001)	0.84 (0.000)	0.27
	h=5	none	79.37 (0.139)	2.07 (0.005)	2.65 (0.001)	0.38
		BC	83.66 (0.131)	2.29 (0.006)	2.65 (0.001)	0.25
		RF	85.60 (0.113)	2.29 (0.006)	2.65 (0.001)	0.24
	h=10	none	71.74 (0.170)	2.68 (0.009)	3.89 (0.002)	0.56
		BC	79.31 (0.172)	3.31 (0.012)	3.89 (0.002)	0.31
		RF	83.17 (0.144)	3.33 (0.011)	3.89 (0.002)	0.27
Y5	h=1	none	88.57 (0.068)	1.50 (0.002)	1.65 (0.001)	0.16
		BC	89.17 (0.065)	1.52 (0.002)	1.65 (0.001)	0.14
		RF	89.61 (0.062)	1.52 (0.002)	1.65 (0.001)	0.13
	h=5	none	80.03 (0.134)	4.42 (0.011)	5.55 (0.002)	0.36
		BC	84.34 (0.124)	4.90 (0.012)	5.55 (0.002)	0.23
		RF	86.08 (0.108)	4.89 (0.012)	5.55 (0.002)	0.21
	h=10	none	72.55 (0.165)	5.64 (0.018)	8.08 (0.003)	0.54
		BC	80.08 (0.164)	6.97 (0.024)	8.08 (0.003)	0.29
		RF	83.70 (0.136)	7.02 (0.023)	8.08 (0.003)	0.25
Y10	h=1	none	90.11 (0.074)	1.13 (0.002)	1.14 (0.000)	0.06
		BC	90.67 (0.070)	1.14 (0.002)	1.14 (0.000)	0.05
		RF	90.95 (0.067)	1.14 (0.002)	1.14 (0.000)	0.05
	h=5	none	81.74 (0.136)	2.85 (0.007)	3.35 (0.001)	0.29
		BC	85.80 (0.123)	3.17 (0.008)	3.35 (0.001)	0.15
		RF	86.55 (0.112)	3.15 (0.008)	3.35 (0.001)	0.15
	h=10	none	77.00 (0.151)	3.33 (0.010)	4.37 (0.002)	0.43
		BC	82.93 (0.149)	4.10 (0.014)	4.37 (0.002)	0.19
		RF	84.97 (0.125)	4.13 (0.014)	4.37 (0.002)	0.16
Y25	h=1	none	90.88 (0.070)	1.66 (0.003)	1.64 (0.001)	0.06
		BC	91.42 (0.066)	1.68 (0.003)	1.64 (0.001)	0.06
		RF	91.66 (0.064)	1.68 (0.003)	1.64 (0.001)	0.06
	h=5	none	81.76 (0.134)	4.28 (0.011)	5.06 (0.002)	0.29
		BC	85.85 (0.121)	4.77 (0.012)	5.06 (0.002)	0.15
		RF	86.75 (0.109)	4.74 (0.012)	5.06 (0.002)	0.15
	h=10	none	76.47 (0.154)	5.08 (0.015)	6.72 (0.003)	0.44
		BC	82.62 (0.151)	6.26 (0.021)	6.72 (0.003)	0.20
		RF	84.93 (0.126)	6.31 (0.021)	6.72 (0.003)	0.17

Table C2. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created with both common factors following AR(2) processes with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 50$. Nominal coverage 95%. IC_3 used in the estimation of R, AICc used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	87.43 (0.078)	0.95 (0.001)	1.05 (0.000)	0.18
		BC	88.07 (0.073)	0.96 (0.002)	1.05 (0.000)	0.16
		RF	88.47 (0.070)	0.96 (0.002)	1.05 (0.000)	0.16
	h=5	none	79.30 (0.140)	2.64 (0.006)	3.29 (0.001)	0.36
		BC	83.75 (0.128)	2.93 (0.007)	3.29 (0.001)	0.23
		RF	85.26 (0.111)	2.91 (0.007)	3.29 (0.001)	0.22
	h=10	none	72.43 (0.166)	3.29 (0.010)	4.61 (0.002)	0.52
		BC	79.80 (0.164)	4.06 (0.014)	4.61 (0.002)	0.28
		RF	83.05 (0.135)	4.05 (0.013)	4.61 (0.002)	0.25
Y2	h=1	none	83.66 (0.082)	0.68 (0.001)	0.84 (0.000)	0.31
		BC	84.39 (0.081)	0.69 (0.001)	0.84 (0.000)	0.30
		RF	84.89 (0.076)	0.69 (0.001)	0.84 (0.000)	0.29
	h=5	none	78.06 (0.141)	2.03 (0.005)	2.65 (0.001)	0.41
		BC	82.65 (0.133)	2.26 (0.006)	2.65 (0.001)	0.28
		RF	84.60 (0.116)	2.25 (0.005)	2.65 (0.001)	0.26
	h=10	none	70.13 (0.174)	2.64 (0.009)	3.88 (0.002)	0.58
		BC	78.04 (0.177)	3.27 (0.012)	3.88 (0.002)	0.34
		RF	81.97 (0.148)	3.26 (0.011)	3.88 (0.002)	0.30
Y5	h=1	none	87.64 (0.075)	1.48 (0.002)	1.65 (0.001)	0.18
		BC	88.29 (0.072)	1.50 (0.002)	1.65 (0.001)	0.16
		RF	88.76 (0.067)	1.50 (0.002)	1.65 (0.001)	0.15
	h=5	none	78.73 (0.140)	4.36 (0.011)	5.55 (0.002)	0.39
		BC	83.23 (0.130)	4.84 (0.012)	5.55 (0.002)	0.25
		RF	85.09 (0.113)	4.82 (0.012)	5.55 (0.002)	0.24
	h=10	none	70.98 (0.171)	5.60 (0.018)	8.07 (0.003)	0.56
		BC	78.76 (0.171)	6.92 (0.024)	8.07 (0.003)	0.31
		RF	82.49 (0.142)	6.90 (0.023)	8.07 (0.003)	0.28
Y10	h=1	none	89.10 (0.081)	1.11 (0.002)	1.14 (0.000)	0.09
		BC	89.64 (0.078)	1.12 (0.002)	1.14 (0.000)	0.07
		RF	89.98 (0.074)	1.12 (0.002)	1.14 (0.000)	0.07
	h=5	none	80.35 (0.145)	2.81 (0.007)	3.35 (0.001)	0.32
		BC	84.45 (0.135)	3.13 (0.008)	3.35 (0.001)	0.18
		RF	85.37 (0.120)	3.09 (0.008)	3.35 (0.001)	0.18
	h=10	none	75.51 (0.160)	3.31 (0.010)	4.37 (0.002)	0.45
		BC	81.48 (0.160)	4.06 (0.014)	4.37 (0.002)	0.21
		RF	83.64 (0.133)	4.04 (0.014)	4.37 (0.002)	0.19
Y25	h=1	none	89.76 (0.077)	1.62 (0.003)	1.64 (0.001)	0.06
		BC	90.36 (0.074)	1.64 (0.003)	1.64 (0.001)	0.05
		RF	90.62 (0.070)	1.64 (0.003)	1.64 (0.001)	0.05
	h=5	none	80.14 (0.143)	4.20 (0.010)	5.05 (0.002)	0.32
		BC	84.41 (0.131)	4.68 (0.012)	5.05 (0.002)	0.19
		RF	85.41 (0.118)	4.63 (0.012)	5.05 (0.002)	0.19
	h=10	none	74.61 (0.160)	5.01 (0.015)	6.72 (0.003)	0.47
		BC	81.12 (0.159)	6.16 (0.021)	6.72 (0.003)	0.23
		RF	83.49 (0.132)	6.13 (0.020)	6.72 (0.003)	0.21

Table C3. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created with both common factors following AR(2) processes with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 200$. Nominal coverage 95%. IC_3 used in the estimation of R, BIC used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	92.44 (0.030)	1.01 (0.001)	1.05 (0.000)	0.07
		BC	92.44 (0.030)	1.01 (0.001)	1.05 (0.000)	0.07
		RF	92.47 (0.030)	1.01 (0.001)	1.05 (0.000)	0.07
	h=5	none	92.25 (0.040)	3.15 (0.004)	3.29 (0.001)	0.07
		BC	93.02 (0.036)	3.24 (0.004)	3.29 (0.001)	0.04
		RF	92.89 (0.036)	3.23 (0.004)	3.29 (0.001)	0.04
	h=10	none	90.51 (0.055)	4.25 (0.006)	4.61 (0.002)	0.13
		BC	92.32 (0.048)	4.55 (0.007)	4.61 (0.002)	0.04
		RF	92.29 (0.047)	4.56 (0.007)	4.61 (0.002)	0.04
Y2	h=1	none	90.45 (0.031)	0.75 (0.001)	0.84 (0.000)	0.16
		BC	90.50 (0.030)	0.75 (0.001)	0.84 (0.000)	0.16
		RF	90.52 (0.030)	0.75 (0.001)	0.84 (0.000)	0.16
	h=5	none	92.05 (0.039)	2.51 (0.003)	2.65 (0.001)	0.08
		BC	92.79 (0.036)	2.58 (0.003)	2.65 (0.001)	0.05
		RF	92.74 (0.035)	2.58 (0.003)	2.65 (0.001)	0.05
	h=10	none	90.24 (0.057)	3.55 (0.005)	3.89 (0.002)	0.14
		BC	92.08 (0.051)	3.82 (0.006)	3.89 (0.002)	0.05
		RF	92.20 (0.048)	3.83 (0.006)	3.89 (0.002)	0.04
Y5	h=1	none	93.32 (0.027)	1.62 (0.001)	1.65 (0.001)	0.04
		BC	93.34 (0.026)	1.62 (0.001)	1.65 (0.001)	0.03
		RF	93.37 (0.026)	1.62 (0.001)	1.65 (0.001)	0.03
	h=5	none	92.31 (0.039)	5.31 (0.006)	5.55 (0.002)	0.07
		BC	93.05 (0.036)	5.48 (0.006)	5.55 (0.002)	0.03
		RF	92.97 (0.035)	5.46 (0.006)	5.55 (0.002)	0.04
	h=10	none	90.38 (0.057)	7.42 (0.011)	8.08 (0.003)	0.13
		BC	92.24 (0.050)	7.97 (0.012)	8.08 (0.003)	0.04
		RF	92.32 (0.047)	8.00 (0.012)	8.08 (0.003)	0.04
Y10	h=1	none	93.15 (0.030)	1.13 (0.001)	1.13 (0.000)	0.02
		BC	93.18 (0.030)	1.13 (0.001)	1.13 (0.000)	0.02
		RF	93.15 (0.030)	1.13 (0.001)	1.13 (0.000)	0.02
	h=5	none	92.50 (0.038)	3.24 (0.004)	3.35 (0.001)	0.06
		BC	93.24 (0.035)	3.34 (0.004)	3.35 (0.001)	0.02
		RF	93.05 (0.036)	3.32 (0.004)	3.35 (0.001)	0.03
	h=10	none	91.00 (0.051)	4.07 (0.006)	4.37 (0.002)	0.11
		BC	92.54 (0.047)	4.34 (0.006)	4.37 (0.002)	0.03
		RF	92.37 (0.048)	4.33 (0.006)	4.37 (0.002)	0.04
Y25	h=1	none	93.84 (0.029)	1.67 (0.001)	1.64 (0.001)	0.03
		BC	93.89 (0.028)	1.67 (0.001)	1.64 (0.001)	0.03
		RF	93.87 (0.028)	1.67 (0.001)	1.64 (0.001)	0.03
	h=5	none	92.52 (0.038)	4.89 (0.006)	5.05 (0.002)	0.06
		BC	93.21 (0.035)	5.03 (0.006)	5.05 (0.002)	0.02
		RF	93.08 (0.036)	5.02 (0.006)	5.05 (0.002)	0.03
	h=10	none	90.92 (0.052)	6.24 (0.009)	6.72 (0.003)	0.11
		BC	92.51 (0.047)	6.67 (0.010)	6.72 (0.003)	0.03
		RF	92.40 (0.047)	6.67 (0.010)	6.72 (0.003)	0.03

Table C4. Results of Monte Carlo simulation, 10,000 replications. Five representative time series created with both common factors following AR(2) processes with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 200$. Nominal coverage 95%. IC_3 used in the estimation of R, AICc used in the selection of $\hat{p}$.

Series	Horizon	Correction	C_m (se)	L_m (se)	L_t (se)	CQ_m
Y1	h=1	none	92.03 (0.032)	1.00 (0.001)	1.05 (0.000)	0.08
		BC	92.08 (0.031)	1.00 (0.001)	1.05 (0.000)	0.08
		RF	92.06 (0.032)	1.00 (0.001)	1.05 (0.000)	0.08
	h=5	none	91.98 (0.042)	3.14 (0.004)	3.29 (0.001)	0.08
		BC	92.76 (0.038)	3.23 (0.004)	3.29 (0.001)	0.04
		RF	92.66 (0.037)	3.22 (0.004)	3.29 (0.001)	0.05
	h=10	none	90.36 (0.058)	4.26 (0.006)	4.61 (0.002)	0.12
		BC	92.11 (0.051)	4.55 (0.007)	4.61 (0.002)	0.04
		RF	92.05 (0.049)	4.54 (0.007)	4.61 (0.002)	0.05
Y2	h=1	none	89.99 (0.033)	0.74 (0.001)	0.84 (0.000)	0.17
		BC	90.06 (0.032)	0.74 (0.001)	0.84 (0.000)	0.17
		RF	90.13 (0.032)	0.74 (0.001)	0.84 (0.000)	0.17
	h=5	none	91.73 (0.041)	2.49 (0.003)	2.65 (0.001)	0.09
		BC	92.53 (0.038)	2.57 (0.003)	2.65 (0.001)	0.06
		RF	92.46 (0.036)	2.56 (0.003)	2.65 (0.001)	0.06
	h=10	none	90.04 (0.060)	3.55 (0.005)	3.88 (0.002)	0.14
		BC	91.86 (0.054)	3.81 (0.006)	3.88 (0.002)	0.05
		RF	91.92 (0.049)	3.80 (0.006)	3.88 (0.002)	0.05
Y5	h=1	none	92.90 (0.028)	1.60 (0.001)	1.65 (0.001)	0.05
		BC	92.97 (0.028)	1.61 (0.001)	1.65 (0.001)	0.05
		RF	92.98 (0.028)	1.61 (0.001)	1.65 (0.001)	0.05
	h=5	none	92.01 (0.041)	5.29 (0.006)	5.55 (0.002)	0.08
		BC	92.78 (0.037)	5.45 (0.006)	5.55 (0.002)	0.04
		RF	92.69 (0.037)	5.43 (0.006)	5.55 (0.002)	0.05
	h=10	none	90.21 (0.060)	7.44 (0.011)	8.07 (0.003)	0.13
		BC	92.03 (0.053)	7.96 (0.012)	8.07 (0.003)	0.05
		RF	92.05 (0.049)	7.94 (0.012)	8.07 (0.003)	0.05
Y10	h=1	none	92.85 (0.032)	1.12 (0.001)	1.14 (0.000)	0.03
		BC	92.89 (0.032)	1.12 (0.001)	1.14 (0.000)	0.03
		RF	92.86 (0.032)	1.12 (0.001)	1.14 (0.000)	0.03
	h=5	none	92.32 (0.040)	3.25 (0.004)	3.35 (0.001)	0.06
		BC	93.03 (0.038)	3.34 (0.004)	3.35 (0.001)	0.03
		RF	92.86 (0.039)	3.32 (0.004)	3.35 (0.001)	0.03
	h=10	none	90.86 (0.054)	4.09 (0.006)	4.37 (0.002)	0.11
		BC	92.36 (0.050)	4.35 (0.007)	4.37 (0.002)	0.03
		RF	92.17 (0.050)	4.33 (0.007)	4.37 (0.002)	0.04
Y25	h=1	none	93.54 (0.031)	1.66 (0.001)	1.64 (0.001)	0.03
		BC	93.59 (0.030)	1.66 (0.001)	1.64 (0.001)	0.03
		RF	93.59 (0.030)	1.67 (0.001)	1.64 (0.001)	0.03
	h=5	none	92.34 (0.040)	4.89 (0.006)	5.05 (0.002)	0.06
		BC	93.05 (0.037)	5.03 (0.006)	5.05 (0.002)	0.03
		RF	92.87 (0.038)	5.00 (0.006)	5.05 (0.002)	0.03
	h=10	none	90.77 (0.054)	6.27 (0.009)	6.72 (0.003)	0.11
		BC	92.38 (0.048)	6.68 (0.010)	6.72 (0.003)	0.03
		RF	92.15 (0.049)	6.65 (0.010)	6.72 (0.003)	0.04

Table C5. Comparison of relative frequencies in the estimation of $\hat{p}$ by BIC and AICc. The values correspond to a Monte Carlo simulation with 10,000 replications. Two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 50$. Nominal coverage 95%.

Factor	$\hat{p} = 1$	$\hat{p} = 2$	$\hat{p} = 3$	$\hat{p} = 4$	$\hat{p} = 5$	$\hat{p} = 6$
BIC						
F1	4.84	64.87	11.91	7.32	5.64	5.42
F2	10.09	59.03	13.18	7.46	5.04	5.20
AICc						
F1	1.24	37.05	15.65	13.51	13.58	18.97
F2	2.70	34.17	16.71	13.86	13.81	18.75

Table C6. Comparison of relative frequencies in the estimation of $\hat{p}$ by BIC and AICc. The values correspond to a Monte Carlo simulation with 10,000 replications. Two common factors, both following AR(2) models with normal errors. Model with coefficients $\phi_1^{F1} = 1.475, \phi_2^{F1} = -0.4875, \phi_1^{F2} = 1.4, \phi_2^{F2} = -0.45$. $T = 200$. Nominal coverage 95%.

Factor	$\hat{p} = 1$	$\hat{p} = 2$	$\hat{p} = 3$	$\hat{p} = 4$	$\hat{p} = 5$	$\hat{p} = 6$
BIC						
F1	0.00	89.33	7.56	2.07	0.77	0.27
F2	0.03	85.96	10.44	2.39	0.77	0.41
AICc						
F1	0.00	36.91	16.35	14.14	13.68	18.92
F2	0.00	32.78	18.89	14.00	13.98	20.35

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